

# Power of Quantum Computation with Few Clean Qubits

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## Abstract

This paper investigates the power of polynomial-time quantum computation in which only a very limited number of qubits are initially clean in the  $|0\rangle$  state, and all the remaining qubits are initially in the totally mixed state. No initializations of qubits are allowed during the computation, nor intermediate measurements. The main results of this paper are unexpectedly strong error-reducible properties of such quantum computations. It is proved that any problem solvable by a polynomial-time quantum computation with one-sided bounded error that uses logarithmically many clean qubits can also be solvable with exponentially small one-sided error using just two clean qubits, and with polynomially small one-sided error using just one clean qubit (which in particular implies the solvability with any small constant one-sided error). It is further proved in the case of two-sided bounded error that any problem solvable by such a computation with a constant gap between completeness and soundness using logarithmically many clean qubits can also be solvable with exponentially small two-sided error using just two clean qubits. If only one clean qubit is available, the problem is again still solvable with exponentially small error in one of the completeness and soundness and polynomially small error in the other. As an immediate consequence of the above result for the two-sided-error case, it follows that the TRACE ESTIMATION problem defined with *fixed* constant threshold parameters is complete for  $BQ_{\log}P$  and  $BQ_{[1]}P$ , the classes of problems solvable by polynomial-time quantum computations with completeness  $2/3$  and soundness  $1/3$  using logarithmically many clean qubits and just one clean qubit, respectively. The techniques used for proving the error-reduction results may be of independent interest in themselves, and one of the technical tools can also be used to show the hardness of weak classical simulations of one-clean-qubit computations (i.e., DQC1 computations).

# 1 Introduction

## 1.1 Background

An inherent nature of randomized and quantum computing is that the outcome of a computation is probabilistic and may not always be correct. Error reduction, or success-probability amplification, is thus one of the most fundamental issues in randomized and quantum computing. Computation error can be efficiently reduced to be negligibly small in many standard computation models via a simple repetition followed by an OR-type, or an AND-type, or a threshold-value decision, depending on whether the error can happen in the original computation only for yes-instances, or only for no-instances, or for both. Typical examples are polynomial-time randomized and quantum computations with bounded error, and in particular, the error can be made exponentially small in BPP and BQP both in completeness and in soundness, which provides a reasonable ground for the well-used definitions of BPP and BQP that employ bounds  $2/3$  and  $1/3$  for completeness and soundness, respectively. In many other computation models, however, it is unclear whether the error can be reduced efficiently by the simple repetition-based method mentioned above, and more generally, whether error reduction itself is possible. Such a situation often occurs when a computation model involves communications with some untrusted parties, like interactive proof systems. For instance, the simple repetition-based method does work for quantum interactive proofs in the one-sided error case of perfect completeness, but its proof is highly nontrivial [KW00, Gut09]. Moreover, a negative evidence is known in the two-sided error case that the error may not be reduced efficiently via the above-mentioned simple method of repetition with threshold-value decision [MW12], although error reduction itself is anyway possible in this case, as any two-sided-error quantum interactive proof system can be made to have perfect completeness by adding more communication turns [KW00, KLG15] (and the number of communication turns then can be reduced to three). Another situation where error reduction becomes nontrivial (and sometimes impossible) appears when a computation model can use only very limited computational resources, like space-bounded computations. If the resources are too limited, it is simply impossible to repeat the original computation sufficiently many times, which becomes a quite enormous obstacle to error reduction in the case of space-bounded quantum computations when initializations of qubits are disallowed after the computation starts. Indeed, it is known impossible in the case of one-way quantum finite state automata to reduce computation error smaller than some constant [AF98]. Also, it is unclear whether error reduction is possible or not in various logarithmic-space quantum computations. For computations of one-sided bounded error performed by logarithmic-space quantum Turing machines, Watrous [Wat01] presented a nontrivial method that reduces the error to be exponentially small. Other than this result, error-reduction techniques have not been developed much for space-bounded quantum computations.

Another well-studied model of quantum computing with limited computational resources is the *deterministic quantum computation with one quantum bit (DQC1)*, often mentioned as the *one-clean-qubit model*, which was introduced by Knill and Laflamme [KL98], originally motivated by nuclear magnetic resonance (NMR) quantum information processing. A DQC1 computation over  $w$  qubits starts with the initial state of the totally mixed state except for a single clean qubit, namely,  $|0\rangle\langle 0| \otimes \left(\frac{I}{2}\right)^{\otimes(w-1)}$ . After applying a polynomial-size unitary quantum circuit to this state, only a single output qubit is measured in the computational basis at the end of the computation in order to read out the computation result. The DQC1 model may be viewed as a variant of space-bounded quantum computations in a sense, and is believed not to have full computational power of the standard polynomial-time quantum computation. Indeed, it is known strictly less powerful than the standard polynomial-time quantum computation under some reasonable assumptions [ASV06]. Moreover, since any quantum computation over the totally mixed state  $\left(\frac{I}{2}\right)^{\otimes w}$  is trivially simulatable by a classical computation, the DQC1 model looks easy to classically simulate at first glance. Surprisingly, however, the model turned out to be able to efficiently solve several problems for which no efficient classical algorithms are known, such as calculations of the spectral density [KL98], an integrability tester [PLMP03], the fidelity decay [PBKLO04], Jones and HOMFLY polynomials [SJ08, JW09], and an invariant of 3-manifolds [JA14]. More precisely, DQC1 computations can solve the decisional versions of these problems with two-sided bounded error. Computation error can be quite large in such computations, and in

fact, the gap between completeness and soundness is allowed to be polynomially small. The only method known for amplifying success probability of these computations is to sequentially repeat an attempt of the computation polynomially many times, but this requires the clean qubit to be initialized every time after finishing one attempt, and moreover, the result of each attempt must be recorded to classical work space prepared outside of the DQC1 model. It is definitely more desirable if computation error can be reduced without such initializations. The situation is similar even when the number of clean qubits is allowed to be logarithmically many with respect to the input length. It is also known that any quantum computation of two-sided bounded error that uses logarithmically many clean qubits can be simulated by a quantum computation still of two-sided bounded error that uses just one clean qubit, but the known method for this simulation considerably increases the computational error, and the gap between completeness and soundness becomes polynomially small.

## 1.2 Main Results

This paper develops methods of reducing computation error in quantum computations with few clean qubits, including the DQC1 model. As will be presented below, the methods proposed in this paper are unexpectedly powerful and provide an almost fully satisfying solution in the cases of one-sided bounded error, both to the reducibility of computation error and to the reducibility of the number of clean qubits. In the case of two-sided bounded error, the methods in this paper are applicable only when there is a constant gap between completeness and soundness in the original computation, but still significantly improve the situation of quantum computations with few clean qubits.

Let  $Q_{\log}P(c, s)$ ,  $Q_{[1]}P(c, s)$ , and  $Q_{[2]}P(c, s)$  denote the class of problems solvable by a polynomial-time quantum computation with completeness  $c$  and soundness  $s$  that uses logarithmically many clean qubits, one clean qubit, and two clean qubits, respectively. The rigorous definitions of these complexity classes will be found in Subsection 3.4.

First, in the case of one-sided bounded error, it is proved that any problem solvable by a polynomial-time quantum computation with one-sided bounded error that uses logarithmically many clean qubits can also be solvable by that with exponentially small one-sided error using just two clean qubits.

**Theorem 1.** *For any polynomially bounded function  $p: \mathbb{Z}^+ \rightarrow \mathbb{N}$  and any polynomial-time computable function  $s: \mathbb{Z}^+ \rightarrow [0, 1]$  satisfying  $1 - s \geq \frac{1}{q}$  for some polynomially bounded function  $q: \mathbb{Z}^+ \rightarrow \mathbb{N}$ ,*

$$Q_{\log}P(1, s) \subseteq Q_{[2]}P(1, 2^{-p}).$$

If only one clean qubit is available, the problem is still solvable with polynomially small one-sided error (which in particular implies the solvability with any small constant one-sided error).

**Theorem 2.** *For any polynomially bounded function  $p: \mathbb{Z}^+ \rightarrow \mathbb{N}$  and any polynomial-time computable function  $s: \mathbb{Z}^+ \rightarrow [0, 1]$  satisfying  $1 - s \geq \frac{1}{q}$  for some polynomially bounded function  $q: \mathbb{Z}^+ \rightarrow \mathbb{N}$ ,*

$$Q_{\log}P(1, s) \subseteq Q_{[1]}P\left(1, \frac{1}{p}\right).$$

The above two theorems are for the one-sided error case of perfect completeness, and similar statements hold even for the case of perfect soundness, by considering the complement of the problem.

**Corollary 3.** *For any polynomially bounded function  $p: \mathbb{Z}^+ \rightarrow \mathbb{N}$  and any polynomial-time computable function  $c: \mathbb{Z}^+ \rightarrow [0, 1]$  satisfying  $c \geq \frac{1}{q}$  for some polynomially bounded function  $q: \mathbb{Z}^+ \rightarrow \mathbb{N}$ , the following two properties hold:*

$$(i) \quad Q_{\log}P(c, 0) \subseteq Q_{[2]}P(1 - 2^{-p}, 0),$$

$$(ii) \quad Q_{\log}P(c, 0) \subseteq Q_{[1]}P\left(1 - \frac{1}{p}, 0\right).$$

In the case of two-sided bounded error, similar statements are proved on the condition that there is a constant gap between completeness and soundness in the original computation. Namely, it is proved that any problem solvable by a polynomial-time quantum computation that uses logarithmically many clean qubits and has a constant gap between completeness and soundness can also be solvable by that with exponentially small two-sided error using just two clean qubits.

**Theorem 4.** *For any polynomially bounded function  $p: \mathbb{Z}^+ \rightarrow \mathbb{N}$  and any constants  $c$  and  $s$  in  $\mathbb{R}$  satisfying  $0 < s < c < 1$ ,*

$$Q_{\log}P(c, s) \subseteq Q_{[2]}P(1 - 2^{-p}, 2^{-p}).$$

If only one clean qubit is available, the problem is again still solvable with exponentially small error in one of the completeness and soundness and polynomially small error in the other.

**Theorem 5.** *For any polynomially bounded function  $p: \mathbb{Z}^+ \rightarrow \mathbb{N}$  and any constants  $c$  and  $s$  in  $\mathbb{R}$  satisfying  $0 < s < c < 1$ ,*

$$Q_{\log}P(c, s) \subseteq Q_{[1]}P\left(1 - 2^{-p}, \frac{1}{p}\right) \cap Q_{[1]}P\left(1 - \frac{1}{p}, 2^{-p}\right).$$

The ideas for the proofs of these statements will be overviewed in Section 2. The techniques developed in the proofs may be of independent interest in themselves, and one of the technical tools can be used to show the hardness of weak classical simulations of DQC1 computations, as will be summarized below.

### 1.3 Further Results

**Completeness results for TRACE ESTIMATION problem** Define the complexity classes  $BQ_{\log}P$  and  $BQ_{[1]}P$  by  $BQ_{\log}P = Q_{\log}P(\frac{2}{3}, \frac{1}{3})$  and  $BQ_{[1]}P = Q_{[1]}P(\frac{2}{3}, \frac{1}{3})$ , respectively. An immediate consequence of Theorem 5 is that the TRACE ESTIMATION problem is complete for  $BQ_{\log}P$  and  $BQ_{[1]}P$  under polynomial-time many-one reduction, even when the problem is defined with *fixed* constant parameters that specify the bounds on normalized traces in the yes-instance and no-instance cases.

Given a description of a quantum circuit that specifies a unitary transformation  $U$ , the TRACE ESTIMATION problem specified with two parameters  $a$  and  $b$  satisfying  $-1 \leq b < a \leq 1$  is the problem of deciding whether the real part of the normalized trace of  $U$  is at least  $a$  or it is at most  $b$ .

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#### TRACE ESTIMATION PROBLEM: $\text{TREST}(a, b)$

**Input:** A description of a quantum circuit  $Q$  that implements a unitary transformation  $U$  over  $n$  qubits.

**Yes Instances:**  $\frac{1}{2^n} \Re(\text{tr } U) \geq a$ .

**No Instances:**  $\frac{1}{2^n} \Re(\text{tr } U) \leq b$ .

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The paper by Knill and Laflamme [KL98] that introduced the DQC1 model already pointed out that this problem is closely related to the DQC1 computation. This point was further clarified in the succeeding literature (see Refs. [She06, She09, SJ08], for instance). More precisely, consider a variant of the TRACE ESTIMATION problem where the two parameters  $a$  and  $b$  may depend on the input length (i.e., the length of the description of  $Q$ ). It is known that this version of the TRACE ESTIMATION problem, for any  $a$  and  $b$  such that the gap  $a - b$  is bounded from below by an inverse-polynomial with respect to the input length, can be solved by a DQC1 computation with *some* two-sided bounded error where the completeness and soundness parameters  $c$  and  $s$  depend on  $a$  and  $b$ . It

is also known that, for any two nonnegative parameters  $a$  and  $b$  such that the gap  $a - b$  is bounded from below by an inverse-polynomial with respect to the input length, the corresponding version of the TRACE ESTIMATION problem is hard for the complexity class  $Q_{[1]}P(c, s)$  for *some* completeness parameter  $c$  and soundness parameter  $s$  that depend on  $a$  and  $b$ . Hence, the TRACE ESTIMATION problem essentially characterizes the power of the DQC1 computation, except the following subtle point. One thing to be pointed out in the existing arguments above is that, when the parameters  $a$  and  $b$  are fixed for the TRACE ESTIMATION problem, the completeness  $c$  and soundness  $s$  with which the problem is in  $Q_{[1]}P(c, s)$  are *different* from the completeness  $c'$  and soundness  $s'$  with which the problem is hard for  $Q_{[1]}P(c', s')$ . Namely, given two nonnegative parameters  $a$  and  $b$  of the problem, the computation solves the problem with completeness  $c = (1 + a)/2$  and soundness  $s = (1 + b)/2$ , while the problem is hard for the class with completeness  $c' = a/4$  and soundness  $s' = b/4$ . Therefore, the existing arguments are slightly short for proving  $BQ_{[1]}P$ -completeness of the TRACE ESTIMATION problem with fixed parameters  $a$  and  $b$  (and  $Q_{[1]}P(c, s)$ -completeness of that for fixed completeness and soundness parameters  $c$  and  $s$ , in general).

In contrast, with Theorem 5 in hand, it is immediate to show that the TRACE ESTIMATION problem is complete for  $BQ_{\log}P$  and for  $BQ_{[1]}P$  for any constants  $a$  and  $b$  satisfying  $0 < b < a < 1$ .

**Theorem 6.** *For any constants  $a$  and  $b$  in  $\mathbb{R}$  satisfying  $0 < b < a < 1$ ,  $\text{TREST}(a, b)$  is complete for  $BQ_{\log}P$  and for  $BQ_{[1]}P$  under polynomial-time many-one reduction.*

**Hardness of weak classical simulations of DQC1 computation** Recently, quite a few number of studies focused on the hardness of *weak* classical simulations of restricted models of quantum computing under some reasonable assumptions [TD04, BJS11, AA13, NVdN13, JVDN14, MFF14, TYT14, Bro15, TTYT15]. Namely, a plausible assumption in complexity theory leads to the impossibility of efficient sampling by a classical computer according to an output probability distribution generatable with a quantum computing model. Among them are the IQP model [BJS11] and the Boson sampling [AA13], both of which are proved hard for classical computers to simulate within multiplicative error, unless the polynomial-time hierarchy collapses to the third level (in fact, the main result of Ref. [AA13] is a much more meaningful hardness result on the weak simulatability of the Boson sampling within *polynomially small additive error*, but which needs a much stronger complexity assumption than the collapse of polynomial-time hierarchy).

An interesting question to ask is whether a similar result holds even for the DQC1 model. Very recently, Morimae, Fujii, and Fitzsimons [MFF14] approached to answering the question. They focused on the  $\text{DQC1}_m$ -type computation, the generalization of the DQC1 model that allows  $m$  output qubits to be measured at the end of the computation, and proved that a  $\text{DQC1}_m$ -type computation with  $m \geq 3$  cannot be simulated within multiplicative error unless the polynomial-time hierarchy collapses to the third level. Their proof essentially shows that any  $\text{PostBQP}$  circuit can be simulated by a  $\text{DQC1}_3$ -type computation, where  $\text{PostBQP}$  is the complexity class corresponding to bounded-error quantum polynomial-time computations with postselection, which is known equivalent to  $\text{PP}$  [Aar05]. By an argument similar to that in Ref. [BJS11], it follows that  $\text{PP}$  is in  $\text{PostBPP}$  (the version of  $\text{BPP}$  with postselection), if the  $\text{DQC1}_3$ -type computation is classically simulatable within multiplicative error. Together with Toda's theorem [Tod91], this implies the collapse of the polynomial-time hierarchy to the third level.

One obvious drawback of the existing argument above is an inevitable postselection measurement inherent to the definition of  $\text{PostBQP}$ . This becomes a quite essential obstacle when trying to extend this argument to the DQC1 model, where only one qubit is allowed to be measured.

To deal with the DQC1 model, this paper takes a different approach by considering the complexity class  $\text{NQP}$  introduced in Ref. [ADH97] or the class  $\text{SBQP}$  introduced in Ref. [Kup15]. Let  $\text{NQ}_{[1]}P$  and  $\text{SBQ}_{[1]}P$  be the variants of  $\text{NQP}$  and  $\text{SBQP}$ , respectively, in which the quantum computation performed is restricted to the DQC1 computation (the precise definitions of  $\text{NQ}_{[1]}P$  and  $\text{SBQ}_{[1]}P$  will be found in Subsection 3.4). From one of the technical tools used for proving the main results of this paper (the ONE-CLEAN-QUBIT SIMULATION PROCEDURE in Subsection 4.1), it is almost immediate to show the following theorem that states that the restriction to the DQC1 computation does not change the complexity classes  $\text{NQP}$  and  $\text{SBQP}$ .

**Theorem 7.**  $\text{NQP} = \text{NQ}_{[1]}\text{P}$  and  $\text{SBQP} = \text{SBQ}_{[1]}\text{P}$ .

If any DQC1 computation were classically simulatable within multiplicative error, however, the class  $\text{NQ}_{[1]}\text{P}$  would be included in NP and the class  $\text{SBQ}_{[1]}\text{P}$  would be included in SBP, where SBP is a classical version of SBQP in short, introduced in Ref. [BGM06]. Similarly, if any DQC1 computation were classically simulatable within exponentially small additive error, both  $\text{NQ}_{[1]}\text{P}$  and  $\text{SBQ}_{[1]}\text{P}$  would be included in SBP. Combined with Theorem 7, any of the inclusions  $\text{NQ}_{[1]}\text{P} \subseteq \text{NP}$ ,  $\text{SBQ}_{[1]}\text{P} \subseteq \text{SBP}$ , and  $\text{NQ}_{[1]}\text{P} \subseteq \text{SBP}$  further implies an implausible consequence that  $\text{PH} = \text{AM}$ , which in particular implies the collapse of the polynomial-time hierarchy to the second level. Accordingly, the following theorem holds.

**Theorem 8.** *The DQC1 model is not classically simulatable either within multiplicative error or exponentially small additive error, unless  $\text{PH} = \text{AM}$ .*

## 1.4 Organization of the Paper

Section 2 overviews the proofs of the error reduction results, the main results of this paper. Section 3 summarizes the notions and properties that are used throughout this paper. Section 4 provides detailed descriptions and rigorous analyses of three key technical tools used in this paper. Section 5 then rigorously proves Theorems 1 and 2 and Corollary 3, the error reduction results in the cases of one-sided bounded error. Section 6 treats the two-sided-bounded-error cases, by first providing one more technical tool, and then rigorously proving Theorems 4 and 5. The completeness results (Theorem 6) and the results related to the hardness of weak classical simulations of the DQC1 model (Theorems 7 and 8) are proved in Sections 7 and 8, respectively. Finally, Section 9 concludes the paper with some open problems.

## 2 Overview of Error Reduction Results

This section presents an overview of the proofs for the error reduction results. First, Subsection 2.1 provides high-level descriptions of the proofs of Theorems 1 and 2, the theorems for the one-sided error case of perfect completeness. Compared with the two-sided-error case, the proof construction is relatively simpler in the perfect-completeness case, but already involves most of key technical ingredients of this paper. Subsection 2.2 then explains the further idea that proves Theorems 4 and 5, the theorems for the two-sided-error case.

### 2.1 Proof Ideas of Theorems 1 and 2

Let  $A = (A_{\text{yes}}, A_{\text{no}})$  be any problem in  $\text{Q}_{\log}\text{P}(1, s)$ , where the function  $s$  defining the soundness is bounded away from one by an inverse-polynomial, and consider a polynomial-time uniformly generated family of quantum circuits that puts  $A$  in  $\text{Q}_{\log}\text{P}(1, s)$ . Let  $Q_x$  denote the quantum circuit from this family when the input is  $x$ , where  $Q_x$  acts over  $w(|x|)$  qubits for some polynomially bounded function  $w$ , and is supposed to be applied to the initial state  $(|0\rangle\langle 0|)^{\otimes k(|x|)} \otimes \left(\frac{I}{2}\right)^{\otimes (w(|x|) - k(|x|))}$  that contains exactly  $k(|x|)$  clean qubits, for some logarithmically bounded function  $k$ .

Theorems 1 and 2 are proved by constructing circuits with desirable properties from the original circuit  $Q_x$ . The construction is essentially the same for both of the two theorems and consists of three stages of transformations of circuits: The first stage reduces the number of necessary clean qubits to just one, while keeping perfect completeness and soundness still bounded away from one by an inverse-polynomial. The second stage then makes the acceptance probability of no-instances arbitrarily close to  $1/2$ , still using just one clean qubit and keeping perfect completeness. Here, it not only makes the soundness (i.e., the upper bound of the acceptance probability of no-instances) close to  $1/2$ , but also makes the acceptance probability of no-instances *at least*  $1/2$ . Finally, in the case of Theorem 2, the third stage further reduces soundness error to be polynomially small with the use of just one clean qubit, while preserving the perfect completeness property. If one more clean qubit is available, the third

stage can make soundness exponentially small with keeping perfect completeness, which leads to Theorem 1. The analyses of the third stage effectively use the fact that the acceptance probability of no-instances is close to  $1/2$  after the transformation of the second stage. The rest of this subsection sketches the ideas that realize each of these three stages.

**ONE-CLEAN-QUBIT SIMULATION PROCEDURE** The first stage uses a procedure called the ONE-CLEAN-QUBIT SIMULATION PROCEDURE. Given the quantum circuit  $Q_x$  with a specification of the number  $k(|x|)$  of clean qubits, this procedure results in a quantum circuit  $R_x$  such that the input state to  $R_x$  is supposed to contain just one clean qubit, and when applied to the one-clean-qubit initial state, the acceptance probability of  $R_x$  is still one if  $x$  is in  $A_{\text{yes}}$ , while it is at most  $1 - \delta(|x|)$  if  $x$  is in  $A_{\text{no}}$ , where  $\delta$  is an inverse-polynomial function determined by  $\delta = 2^{-k}(1 - s)$ . It is stressed that the ONE-CLEAN-QUBIT SIMULATION PROCEDURE preserves perfect completeness, which is in stark contrast to the straightforward method of one-clean-qubit simulation.

Consider the  $k(|x|)$ -clean-qubit computation performed with  $Q_x$ . Let  $Q$  denote the quantum register consisting of the  $k(|x|)$  initially clean qubits, and let  $R$  denote the quantum register consisting of the remaining  $w(|x|) - k(|x|)$  qubits that are initially in the totally mixed state. Further let  $Q^{(1)}$  denote the single-qubit quantum register consisting of the first qubit of  $Q$ , which corresponds to the output qubit of  $Q_x$ . In the one-clean-qubit simulation of  $Q_x$  by  $R_x$ , the  $k(|x|)$  qubits in  $Q$  are supposed to be in the totally mixed state initially and  $R_x$  tries to simulate  $Q_x$  only when  $Q$  initially contains the clean all-zero state. To do so,  $R_x$  uses another quantum register  $O$  consisting of just a single qubit, and this qubit in  $O$  is the only qubit that is supposed to be initially clean.

For ease of explanations, assume for a while that all the qubits in  $Q$  are also initially clean even in the case of  $R_x$ . The key idea in the construction of  $R_x$  is the following simulation of  $Q_x$  that makes use of the phase-flip transformation: The simulation first applies the Hadamard transformation  $H$  to the qubit in  $O$  and then flips the phase if and only if the content of  $O$  is 1 *and* the simulation of  $Q_x$  results in rejection (which is realized by performing  $Q_x$  to  $(Q, R)$  and then applying the controlled- $Z$  transformation to  $(O, Q^{(1)})$ , where the content 1 in  $Q^{(1)}$  is assumed to correspond to the rejection in the original computation by  $Q_x$ ). The simulation further performs the inverse of  $Q_x$  to  $(Q, R)$  and again applies the Hadamard transformation  $H$  to  $O$ . At the end of the simulation, the qubit in  $O$  is measured in the computational basis, with the measurement result 0 corresponding to acceptance. The point is that this phase-flip-based construction provides a quite “faithful” simulation of  $Q_x$ , meaning that the rejection probability of the simulation is polynomially related to the rejection probability of the original computation of  $Q_x$  (and in particular, the simulation never rejects when the original computation never rejects, i.e., the simulation preserves the perfect completeness property).

As mentioned before, all the qubits in  $Q$  are supposed to be in the totally mixed state initially in the one-clean-qubit simulation of  $Q_x$  by  $R_x$ , and  $R_x$  tries to simulate  $Q_x$  only when  $Q$  initially contains the clean all-zero state. To achieve this, each of the applications of the Hadamard transformation is replaced by an application of the controlled-Hadamard transformation so that the Hadamard transformation is applied only when all the qubits in  $Q$  are in state  $|0\rangle$ . By considering the one-clean-qubit computations with the circuit family induced by  $R_x$ , the perfect completeness property is preserved and soundness is still bounded away from one by an inverse-polynomial (although the rejection probability becomes smaller for no-instances by a multiplicative factor of  $2^{-k}$ , where notice that  $2^{-k}$  is an inverse-polynomial as  $k$  is a logarithmically bounded function). The construction of  $R_x$  is summarized in Figure 1. A precise description and a detailed analysis of the ONE-CLEAN-QUBIT SIMULATION PROCEDURE will be presented in Subsection 4.1.

**RANDOMNESS AMPLIFICATION PROCEDURE** The second stage uses the procedure called the RANDOMNESS AMPLIFICATION PROCEDURE. Given the circuit  $R_x$  constructed in the first stage, this procedure results in a quantum circuit  $R'_x$  such that the input state to  $R'_x$  is still supposed to contain just one clean qubit, and when applied to the one-clean-qubit initial state, the acceptance probability of  $R'_x$  is still one if  $x$  is in  $A_{\text{yes}}$ , while it is in the interval  $[\frac{1}{2}, \frac{1}{2} + \varepsilon(|x|)]$  if  $x$  is in  $A_{\text{no}}$  for some sufficiently small function  $\varepsilon$ .

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### ONE-CLEAN-QUBIT SIMULATION PROCEDURE — Simplified Description

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1. Prepare a single-qubit register O, a  $k(|x|)$ -qubit register Q, and a  $(w(|x|) - k(|x|))$ -qubit register R, where the qubit in O is supposed to be initially in state  $|0\rangle$ , while all the qubits in Q and R are supposed to be initially in the totally mixed state  $I/2$ .  
Apply  $H$  to O if all the qubits in Q are in state  $|0\rangle$ .
  2. Apply  $Q_x$  to (Q, R).
  3. Apply the phase-flip (i.e., multiply the phase by  $-1$ ) if the content of  $(O, Q^{(1)})$  is 11, where  $Q^{(1)}$  denotes the single-qubit register consisting of the first qubit of Q.
  4. Apply  $Q_x^\dagger$  to (Q, R).
  5. Apply  $H$  to O if all the qubits in Q are in state  $|0\rangle$ . Measure the qubit in O in the computational basis. Accept if this results in  $|0\rangle$ , and reject otherwise.
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Figure 1: The ONE-CLEAN-QUBIT SIMULATION PROCEDURE induced by a quantum circuit  $Q_x$  with the specification of the number  $k(|x|)$  of clean qubits used in the computation of  $Q_x$  to be simulated (a slightly simplified description).

Consider the one-clean-qubit computation performed with  $R_x$ . Let O denote the single-qubit quantum register consisting of the initially clean qubit, which is also the output qubit of  $R_x$ . Let R denote the quantum register consisting of all the remaining qubits that are initially in the totally mixed state (by the construction of  $R_x$ , R consists of  $w(|x|)$  qubits).

Suppose that the qubit in O is measured in the computational basis after  $R_x$  is applied to the one-clean-qubit initial state  $|0\rangle\langle 0| \otimes (\frac{I}{2})^{\otimes w(|x|)}$  in (O, R). Obviously from the property of  $R_x$ , the measurement results in 0 with probability exactly equal to the acceptance probability  $p_{\text{acc}}$  of the one-clean-qubit computation with  $R_x$ . Now suppose that  $R_x$  is applied to a slightly different initial state  $|1\rangle\langle 1| \otimes (\frac{I}{2})^{\otimes w(|x|)}$  in (O, R), where O initially contains  $|1\rangle$  instead of  $|0\rangle$  and all the qubits in R are again initially in the totally mixed state. The key property here to be proved is that, in this case, the measurement over the qubit in O in the computational basis results in 1 again with probability exactly  $p_{\text{acc}}$ , the acceptance probability of the one-clean-qubit computation with  $R_x$ . This implies that, after the application of  $R_x$  to (O, R) with all the qubits in R being in the totally mixed state, the content of O remains the same with probability exactly  $p_{\text{acc}}$ , and is flipped with probability exactly  $1 - p_{\text{acc}}$ , the rejection probability of the original one-clean-qubit computation with  $R_x$ , regardless of the initial content of O.

The above observation leads to the following construction of the circuit  $R'_x$ . The construction of  $R'_x$  is basically a sequential repetition of the original circuit  $R_x$ . The number  $N$  of repetitions is polynomially many with respect to the input length  $|x|$ , and the point is that the register O is reused for each repetition, and only the qubits in R are refreshed after each repetition (by preparing  $N$  registers  $R_1, \dots, R_N$ , each of which consists of  $w(|x|)$  qubits, the same number of qubits as R, all of which are initially in the totally mixed state). After each repetition the qubit in O is measured in the computational basis (in the actual construction, this step is exactly simulated without any measurement — a single-qubit totally mixed state is prepared as a fresh ancilla qubit for each repetition so that the content of O is copied to this ancilla qubit using the CNOT transformation, and this ancilla qubit is never touched after this CNOT application). Now, no matter which measurement result is obtained at the  $j$ th repetition for every  $j$  in  $\{1, \dots, N\}$ , the register O is reused as it is, and the circuit  $R_x$  is simply applied to (O,  $R_{j+1}$ ) at the  $(j + 1)$ st repetition. After the  $N$  repetitions, the qubit in O is measured in the computational basis, which is the output of  $R'_x$  (the output 0 corresponds to acceptance).



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### RANDOMNESS AMPLIFICATION PROCEDURE — Simplified Description

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1. Prepare a single-qubit register  $O$ , where the qubit in  $O$  is supposed to be initially in state  $|0\rangle$ . Prepare a  $w(|x|)$ -qubit register  $R_j$  for each  $j$  in  $\{1, \dots, N\}$ , where all the qubits in  $R_j$  are supposed to be initially in the totally mixed state  $I/2$ .
  2. For  $j = 1$  to  $N$ , perform the following:
    - 2.1. Apply  $R_x$  to  $(O, R_j)$ .
    - 2.2. Measure the qubit in  $O$  in the computational basis.
  3. Accept if the qubit in  $O$  is in state  $|0\rangle$ , and reject otherwise.
- 

Figure 2: The RANDOMNESS AMPLIFICATION PROCEDURE (a slightly simplified description).

The point is that at each repetition, the content of  $O$  is flipped with probability exactly equal to the rejection probability of the original one-clean-qubit computation of  $R_x$ . Taking into account that  $O$  is initially in state  $|0\rangle$ , the computation of  $R'_x$  results in acceptance if and only if the content of  $O$  is flipped even number of times during the  $N$  repetitions. An analysis on Bernoulli trials then shows that, when the acceptance probability of the original one-clean-qubit computation of  $R_x$  was in the interval  $[\frac{1}{2}, 1)$ , the acceptance probability of the one-clean-qubit computation of  $R'_x$  is at least  $1/2$  and converges linearly to  $1/2$  with respect to the repetition number. On the other hand, when the acceptance probability of the original  $R_x$  was one, the content of  $O$  is never flipped during the computation of  $R'_x$ , and thus the acceptance probability of  $R'_x$  remains one. Figure 2 summarizes the construction of  $R'_x$ . A precise description and a detailed analysis of the RANDOMNESS AMPLIFICATION PROCEDURE will be presented in Subsection 4.2.

**STABILITY CHECKING PROCEDURES** In the case of Theorem 2, the third stage uses the procedure called the ONE-CLEAN-QUBIT STABILITY CHECKING PROCEDURE. Given the circuit  $R'_x$  constructed in the second stage, this procedure results in a quantum circuit  $R''_x$  such that the input state to  $R''_x$  is still supposed to contain just one clean qubit, and when applied to the one-clean-qubit initial state, the acceptance probability of  $R''_x$  is still one if  $x$  is in  $A_{\text{yes}}$ , while it is  $1/p(|x|)$  if  $x$  is in  $A_{\text{no}}$  for a polynomially bounded function  $p$  predetermined arbitrarily.

Consider the one-clean-qubit computation performed with  $R'_x$ . Let  $Q$  denote the single-qubit quantum register consisting of the initially clean qubit, which is also the output qubit of  $R'_x$ . Let  $R$  denote the quantum register consisting of all the remaining qubits that are initially in the totally mixed state, and let  $w'(|x|)$  denote the number of qubits in  $R$ .

Again the key observation is that, after the application of  $R'_x$  to  $(Q, R)$  with all the qubits in  $R$  being in the totally mixed state (followed by the measurement over the qubit in  $Q$  in the computational basis), the content of  $Q$  is flipped with probability exactly equal to the rejection probability of the original one-clean-qubit computation with  $R'_x$ , regardless of the initial content of  $Q$ .

This leads to the following construction of the circuit  $R''_x$ . The construction of  $R''_x$  is again basically a sequential repetition of the original circuit  $R'_x$ , but this time the qubit in  $Q$  is also supposed to be initially in the totally mixed state. The circuit  $R'_x$  is repeatedly applied  $2N$  times, where  $N$  is a power of two and is polynomially many with respect to the input length  $|x|$ , and again the register  $Q$  is reused for each repetition, and only the qubits in  $R$  are refreshed after each repetition (by preparing  $2N$  registers  $R_1, \dots, R_{2N}$ , each of which consists of  $w'(|x|)$  qubits, all of which are initially in the totally mixed state). The key idea for the construction of  $R''_x$  is to use a counter that counts the number of attempts such that the measurement over the qubit in  $Q$  results in  $|1\rangle$  after the

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### ONE-CLEAN-QUBIT STABILITY CHECKING PROCEDURE — Simplified Description

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1. Given a positive integer  $N$  that is a power of two, prepare a counter  $C$  whose value is taken modulo  $2N$ . Choose an integer  $r$  from  $\{0, \dots, N-1\}$  uniformly at random, and initialize a counter  $C$  to  $r$  (which sets the most significant bit of  $C$  to 0). Prepare a single-qubit register  $Q$  and a  $w'(|x|)$ -qubit register  $R_j$  for each  $j$  in  $\{1, \dots, 2N\}$ , where all the qubits in  $Q$  and  $R_j$  are supposed to be initially in the totally mixed state  $I/2$ .
  2. For  $j = 1$  to  $2N$ , perform the following:
    - 2.1. Apply  $R'_x$  to  $(Q, R_j)$ .
    - 2.2. Measure the qubit in  $Q$  in the computational basis. If this results in  $|1\rangle$ , increase the counter  $C$  by one.
  3. Reject if  $N \leq C \leq 2N-1$ , and accept otherwise (i.e., accept iff the most significant bit of  $C$  is 0).
- 

Figure 3: The ONE-CLEAN-QUBIT STABILITY CHECKING PROCEDURE (a simplified description).

application of  $R'_x$  (again each measurement is simulated by a CNOT application using an ancilla qubit of a totally mixed state). Notice that the content of  $Q$  is never flipped regardless of the initial content of  $Q$ , if the original acceptance probability is one in the one-clean-qubit computation with  $R'_x$ . Hence, in this case the counter value either stationarily remains its initial value or is increased exactly by  $2N$ , the number of repetitions. On the other hand, if the original acceptance probability is close to  $1/2$  in the one-clean-qubit computation with  $R'_x$ , the content of  $Q$  is flipped with probability close to  $1/2$  after each application of  $R'_x$  regardless of the initial content of  $Q$ . This means that, after each application of  $R'_x$ , the measurement over the qubit in  $Q$  results in  $|1\rangle$  with probability close to  $1/2$  regardless of the initial content of  $Q$ , and thus, the increment of the counter value must be distributed around  $\frac{1}{2} \cdot 2N = N$  with very high probability. Now, if the counter value is taken modulo  $2N$  and if the unique initially clean qubit is prepared for the most significant bit of the counter (which picks the initial counter value from the set  $\{0, \dots, N-1\}$  uniformly at random), the computational-basis measurement over this most significant qubit of the counter always results in  $|0\rangle$  if  $x$  is in  $A_{\text{yes}}$ , while it results in  $|1\rangle$  with very high probability if  $x$  is in  $A_{\text{no}}$  (which can be made at least  $1 - \frac{1}{p(|x|)}$  for an arbitrarily chosen polynomially bounded function  $p$ , by taking an appropriately large number  $2N$  of the repetition). Figure 3 summarizes the construction of  $R''_x$ .

One drawback of the construction of  $R''_x$  above via the ONE-CLEAN-QUBIT STABILITY CHECKING PROCEDURE is that, in the case of no-instances, there inevitably exist some “bad” initial counter values in  $\{0, \dots, N-1\}$  with which  $R''_x$  is forced to accept with unallowably high probability. For instance, if the initial counter value is 0,  $R''_x$  is forced to accept when the increment of the counter is less than  $N$ , which happens with probability at least a constant. This is the essential reason why the current approach achieves only a polynomially small soundness in the one-clean-qubit case in Theorem 2, as the number of possible initial counter values can be at most polynomially many (otherwise the number of repetitions must be super-polynomially many) and even just one “bad” initial value is problematic to go beyond polynomially small soundness. In contrast, if not just one but two clean qubits are available, one can remove the possibility of “bad” initial counter values, which results in the TWO-CLEAN-QUBIT STABILITY CHECKING PROCEDURE. This time, the circuit  $R'_x$  is repeatedly applied  $8N$  times, and the counter value is taken modulo  $8N$ . The two initially clean qubits are prepared for the most and second-most significant bits of the counter, which results in picking the initial counter value from the set  $\{0, \dots, 2N-1\}$  uniformly at random. Now the point is that the counter value can be increased by  $N$  before the repetition so that the actual initial value of the counter is in  $\{N, \dots, 3N-1\}$ , which discards the tail sets  $\{0, \dots, N-1\}$  and  $\{3N, \dots, 4N-1\}$  of the set  $\{0, \dots, 4N-1\}$ . As the size of the tail sets discarded is sufficiently large, there no longer exists any “bad” initial counter value, which leads to the exponentially small soundness in the two-clean-qubit case in Theorem 1.

Precise descriptions and detailed analyses of the ONE-CLEAN-QUBIT STABILITY CHECKING PROCEDURE and TWO-CLEAN-QUBIT STABILITY CHECKING PROCEDURE will be presented in Subsection 4.3.

## 2.2 Proof Ideas of Theorems 4 and 5

The results for the two-sided error case need more complicated arguments and is proved in eight stages of transformations in total, which are split into three parts.

The first part consists of three stages, and proves that any problem solvable with constant completeness and soundness using logarithmically many clean qubits is also solvable with constant completeness and soundness using just one clean qubit. At the first stage of the first part, by a standard repetition with a threshold-value decision, one first reduces errors to be sufficiently small constants, say, completeness  $15/16$  and soundness  $1/16$ . For this, if the starting computation has a constant gap between completeness and soundness, one requires only a constant number of repetitions, and thus, the resulting computation still requires only logarithmically many clean qubits. The second stage of the first part then reduces the number of clean qubits to just one. The procedure in this stage is exactly the ONE-CLEAN-QUBIT SIMULATION PROCEDURE developed in the first stage of the one-sided error case. The gap between completeness and soundness becomes only an inverse-polynomial by this transformation, but the point is that the gap is still sufficiently larger (i.e. a constant times larger) than the completeness error. Now the third stage of the first part transforms the computation resulting from the second stage to the computation that still uses only one clean qubit and has constant completeness and soundness. The procedure in this stage is exactly the RANDOMNESS AMPLIFICATION PROCEDURE, developed in the second stage of the one-sided error case, and it makes use of the difference of the rates of convergence to  $1/2$  of the acceptance probability between the yes- and no-instance cases. The precise statement corresponding to the first part is found as Lemma 24 in Subsection 6.2.

The second part consists of two stages, and proves that any problem solvable with constant completeness and soundness using just one clean qubit is also solvable with almost-perfect (i.e., exponentially close to one) completeness and soundness below  $1/2$  using just logarithmically many clean qubits. At the first stage of the second part, one reduces both of the completeness and soundness errors to be polynomially small, again by a standard repetition with a threshold-value decision. Note that the computation resulting from the first part requires only one clean qubit. Thus, even when repeated logarithmically many times, the resulting computation uses just logarithmically many clean qubits, and achieves polynomially small errors. The second stage of the second part then repeatedly attempts the computation resulting from the first stage polynomially many times, and accepts if at least one of the attempts results in acceptance (i.e., takes OR of the attempts). A straightforward repetition requires polynomially many clean qubits, and to avoid this problem, after each repetition one tries to recover the clean qubits for reuse by applying the inverse of the computation (the failure of this recovery step is counted as an “acceptance” when taking the OR). This results in a computation that still requires only logarithmically many clean qubits, and has completeness exponentially close to one, while soundness is still below  $1/2$ . The precise statement corresponding to the second part is found as Lemma 25 in Subsection 6.2.

Now the third part is essentially the same as the three-stage transformation of the one-sided error case. From the computation resulting from the second part, the first stage of the third part decreases the number of clean qubits to just one, via the ONE-CLEAN-QUBIT SIMULATION PROCEDURE. The completeness of the resulting computation is still exponentially close to one and its soundness is bounded away from one by an inverse-polynomial. The second stage of the third part then applies the RANDOMNESS AMPLIFICATION PROCEDURE to make the acceptance probability of no-instances arbitrarily close to  $1/2$ , while keeping completeness exponentially close to one. Finally, the third stage of the third part proves that one can further decrease soundness error to be polynomially small using just one qubit via the ONE-CLEAN-QUBIT STABILITY CHECKING PROCEDURE, or to be exponentially small using just two qubits via the TWO-CLEAN-QUBIT STABILITY CHECKING PROCEDURE, while keeping completeness exponentially close to one.

By considering the complement problem, the above argument can also prove the case of exponentially small soundness error in Theorem 5.

### 3 Preliminaries

Throughout this paper, let  $\mathbb{N}$  and  $\mathbb{Z}^+$  denote the sets of positive and nonnegative integers, respectively, and let  $\Sigma = \{0, 1\}$  denote the binary alphabet set. A function  $f: \mathbb{Z}^+ \rightarrow \mathbb{N}$  is *polynomially bounded* if there exists a polynomial-time deterministic Turing machine that outputs  $1^{f(n)}$  on input  $1^n$ . A function  $f: \mathbb{Z}^+ \rightarrow \mathbb{N}$  is *logarithmically bounded* if  $f$  is polynomial-time computable and  $f(n)$  is in  $O(\log n)$ . A function  $f: \mathbb{Z}^+ \rightarrow [0, 1]$  is *negligible* if, for every polynomially bounded function  $g: \mathbb{Z}^+ \rightarrow \mathbb{N}$ , it holds that  $f(n) < 1/g(n)$  for all but finitely many values of  $n$ .

#### 3.1 Useful Properties on Random Variables

This subsection presents two lemmas on properties of the random variable that follows the binomial distribution, which are used in this paper.

The first lemma is a special case of the Hoeffding inequality.

**Lemma 9.** *For any  $n$  in  $\mathbb{N}$  and  $p$  in  $[0, 1]$ , let  $X$  be a random variable over  $\{0, \dots, n\}$  that follows the binomial distribution  $B(n, p)$ . Then, for any  $\delta$  in  $(0, 1)$ ,*

$$\Pr\left[\frac{X}{n} \geq p + \delta\right] < e^{-2\delta^2 n} \quad \text{and} \quad \Pr\left[\frac{X}{n} \leq p - \delta\right] < e^{-2\delta^2 n}.$$

The second lemma is on the probability that a random variable takes an even number when it follows the binomial distribution.

**Lemma 10.** *For any  $n$  in  $\mathbb{N}$  and  $p$  in  $[0, 1]$ , let  $X$  be a random variable over  $\{0, \dots, n\}$  that follows the binomial distribution  $B(n, p)$ . Then,*

$$\Pr[X \text{ is even}] = \frac{1}{2} + \frac{1}{2}(1 - 2p)^n.$$

*Proof.* For each  $j$  in  $\{1, \dots, n\}$ , let  $Y_j$  be an independent random variable over  $\{-1, 1\}$  that takes  $-1$  with probability  $p$ , and let  $Z$  be a random variable over  $\{0, 1\}$  defined by

$$Z = \frac{1}{2} + \frac{1}{2} \prod_{j=1}^n Y_j.$$

Notice that  $Z$  is 1 if and only if there are even number of indices  $j$  such that  $Y_j$  is  $-1$ . Hence,

$$\Pr[X \text{ is even}] = \mathbb{E}[Z] = \frac{1}{2} + \frac{1}{2} \mathbb{E}\left[\prod_{j=1}^n Y_j\right] = \frac{1}{2} + \frac{1}{2} \prod_{j=1}^n \mathbb{E}[Y_j] = \frac{1}{2} + \frac{1}{2}(1 - 2p)^n,$$

where the third equality uses the fact that each  $Y_j$  is an independent random variable, and the claim follows.  $\square$

#### 3.2 Quantum Fundamentals

We assume the reader is familiar with the quantum formalism, including pure and mixed quantum states, density operators, and measurements, as well as the quantum circuit model (see Refs. [NC00, KSV02, Wil13], for instance). This subsection summarizes some notations and properties that are used in this paper.

For every positive integer  $n$ , let  $\mathbb{C}(\Sigma^n)$  denote the  $2^n$ -dimensional complex Hilbert space whose standard basis vectors are indexed by the elements in  $\Sigma^n$ . In this paper, all Hilbert spaces are complex and have dimension a

power of two. For a Hilbert space  $\mathcal{H}$ , let  $I_{\mathcal{H}}$  denote the identity operator over  $\mathcal{H}$ , and let  $\mathbf{U}(\mathcal{H})$  denote the set of unitary operators over  $\mathcal{H}$ . As usual, let

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

denote the Pauli operators, and let

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$$

denote the Hadamard,  $i$ -phase-shift, and  $T$  operators, respectively (the  $T$  operator corresponds to the  $\pi/8$ -gate). Notice that  $S = T^2$  and  $Z = S^2 = T^4$ . For convenience, we may identify a unitary operator with the unitary transformation it induces. In particular, for a unitary operator  $U$ , the induced unitary transformation is also denoted by  $U$ .

For a Hilbert space  $\mathcal{H}$  and a unitary operator  $U$  in  $\mathbf{U}(\mathcal{H})$ , let  $\Lambda(U)$  denote the *controlled- $U$*  operator in  $\mathbf{U}(\mathbb{C}(\Sigma) \otimes \mathcal{H})$  defined by

$$\Lambda(U) = |0\rangle\langle 0| \otimes I_{\mathcal{H}} + |1\rangle\langle 1| \otimes U.$$

For any positive integer  $n \geq 2$ , the  $n$ -*controlled- $U$*  operator in  $\mathbf{U}(\mathbb{C}(\Sigma^n) \otimes \mathcal{H})$  is recursively defined by

$$\Lambda^n(U) = \Lambda(\Lambda^{n-1}(U)) = |0\rangle\langle 0| \otimes I_{\mathbb{C}(\Sigma^{n-1}) \otimes \mathcal{H}} + |1\rangle\langle 1| \otimes \Lambda^{n-1}(U),$$

where  $\Lambda^1(U)$  may be interpreted as the controlled- $U$  operator  $\Lambda(U)$ . In the case where  $U$  is a unitary transformation over a single qubit, notice that the last qubit is the target qubit for each of the unitary transformations  $\Lambda(U)$  and  $\Lambda^n(U)$  in the notation above. For notational convenience, for each positive integer  $n$  and for each integer  $j$  in  $\{1, \dots, n+1\}$ , let  $\Lambda_j^n(U)$  denote the case of the  $n$ -controlled- $U$  operator in which the corresponding transformation uses the  $j$ th qubit as the target qubit. The operator  $\Lambda^n(U)$  corresponds to  $\Lambda_{n+1}^n(U)$  in this notation. The operator  $\Lambda_1^1(U)$  may be simply denoted by  $\Lambda_1(U)$ .

### 3.3 Quantum Circuits

A quantum circuit is specified by a series of quantum gates with designation of qubits to which each quantum gate is applied. It is assumed that any quantum circuit is composed of gates in some reasonable, universal, finite set of quantum gates. A *description* of a quantum circuit is a string in  $\Sigma^*$  that encodes the specification of the quantum circuit. The encoding must be a “reasonable” one, i.e., the number of gates in a circuit encoded is not more than the length of the description of that circuit, and each gate of the circuit is specifiable by a deterministic procedure in time polynomial with respect to the length of the description.

A family  $\{Q_x\}_{x \in \Sigma^*}$  of quantum circuits is *polynomial-time uniformly generated* if there exists a polynomial-time deterministic Turing machine that, on every input  $x$  in  $\Sigma^*$ , outputs a description of  $Q_x$ . For convenience, we may identify a circuit  $Q_x$  with the unitary operator it induces.

*Remark.* Notice that the input  $x$  is “hard-coded” in the generated circuit  $Q_x$  in the above definition of polynomial-time uniformly generated family of quantum circuits, i.e., each circuit  $Q_x$  depends on the input  $x$  itself. The choice of this “hard-coded” definition is just for ease of explanations, and all the results in this paper do remain valid even with a more standard definition of the polynomial-time uniformity where each circuit generated depends only on the input length  $|x|$ , and the input  $x$  is given to the circuit as a read-only input. In fact, all the results still remain valid even when using the *logarithmic-space* uniformly generated family of quantum circuits to define the complexity classes in Subsection 3.4 (by suitably replacing polynomial-time computable functions by logarithmic-space computable functions in some statements and by changing the definitions of polynomially and logarithmically bounded functions by using logarithmic-space deterministic Turing machines and logarithmic-space computable functions). See Subsection 3.5 for further discussions on the uniformity of quantum circuits.

This paper assumes a gate set that includes the Hadamard,  $T$ , and CNOT gates. Note that this assumption is satisfied by many standard gate sets, and is very reasonable and not restrictive. In particular, the gate set proposed in Ref. [BMP<sup>+</sup>00] exactly consists of these three gates. Some useful transformations are in order that are exactly implementable with such a gate set using or not using ancilla qubits:

**Transformations corresponding to Clifford-group operators** First note that, with a gate set satisfying the assumption above, any transformation corresponding to a Clifford-group operator is exactly implementable without using any ancilla qubits.

In particular, the phase-flip transformation  $Z$  is nothing but  $T^4$ , and is easily realized. Thus, the NOT transformation (the operator  $X$ ) is also easily realizable, for  $X = HZH$ .

As  $Z = HXH$ , the controlled- $Z$  transformation  $\Lambda(Z)$  is also realizable by using the decomposition

$$\Lambda(Z) = (I \otimes H) \Lambda(X) (I \otimes H),$$

where  $\Lambda(X)$  is nothing but the CNOT transformation.

**Generalized Toffoli transformations** Using some ancilla qubits, any generalized Toffoli transformation (i.e., the  $n$ -controlled-NOT transformation  $\Lambda^n(X)$  for any positive integer  $n$ ), is also exactly implementable with the gate set discussed, according to the constructions in Ref. [BBC<sup>+</sup>95]. In the construction in Lemma 7.2 of Ref. [BBC<sup>+</sup>95], the number of necessary ancilla qubits grows linearly with respect to the number of control qubits, and the construction in Corollary 7.4 of Ref. [BBC<sup>+</sup>95] uses only two ancilla qubits when  $n \geq 5$ . One very helpful property is that no initializations are required for all these ancilla qubits (and thus, all of them can actually be re-used when applying other generalized Toffoli transformations). In particular, even totally mixed states may be used for these ancilla qubits, and hence, with the above-mentioned gate set, generalized Toffoli transformations may be assumed available freely when constructing quantum circuits that are used for  $k$ -clean-qubit computations defined formally in Subsection 3.4. See Lemma 7.2 and Corollary 7.4 of Ref. [BBC<sup>+</sup>95] for details.

**Controlled-Hadamard transformations** Recall that the Hadamard operator  $H$  is decomposed as

$$H = S^\dagger H T^\dagger X T H S.$$

This implies that the  $n$ -controlled-Hadamard transformation  $\Lambda^n(H)$  for any positive integer  $n$  is decomposed as

$$\Lambda^n(H) = (I^{\otimes n} \otimes S^\dagger) (I^{\otimes n} \otimes H) (I^{\otimes n} \otimes T^\dagger) \Lambda^n(X) (I^{\otimes n} \otimes T) (I^{\otimes n} \otimes H) (I^{\otimes n} \otimes S).$$

As  $S = T^2$ ,  $T^\dagger = T^7$ , and  $S^\dagger = T^6$ , the  $\Lambda^n(H)$  transformation for each  $n$  is exactly implementable by using two Hadamard gates, sixteen  $T$  gates, and one generalized Toffoli transformation (i.e., one  $n$ -controlled-NOT transformation  $\Lambda^n(X)$ ). Clearly, the only necessary ancilla qubits are those used for realizing the  $\Lambda^n(X)$  transformation in this implementation. Hence, provided that generalized Toffoli transformations may be assumed available freely, for all  $n$ ,  $n$ -controlled-Hadamard transformations may also be assumed available freely.

**Increment transformations** For any positive integer  $n$ , let  $U_{+1}(\mathbb{Z}_{2^n})$  denote the increment transformation over  $\mathbb{Z}_{2^n}$ , which is the unitary transformation acting over  $n$  qubits defined by

$$U_{+1}(\mathbb{Z}_{2^n}): |j\rangle \mapsto |(j+1) \bmod 2^n\rangle, \quad \forall j \in \mathbb{Z}_{2^n}$$

Note that

$$U_{+1}(\mathbb{Z}_2) = X,$$

and for each positive integer  $n \geq 2$ ,

$$U_{+1}(\mathbb{Z}_{2^n}) = (I \otimes U_{+1}(\mathbb{Z}_{2^{n-1}})) \Lambda_1^{n-1}(X),$$

where recall that  $\Lambda_1^{n-1}(X)$  corresponds to the  $(n-1)$ -controlled-NOT transformation with the first qubit being the target. Hence, for each positive integer  $n \geq 2$ ,

$$U_{+1}(\mathbb{Z}_{2^n}) = (I^{\otimes(n-1)} \otimes X) (I^{\otimes(n-2)} \otimes \Lambda_1(X)) \cdots (I \otimes \Lambda_1^{n-2}(X)) \Lambda_1^{n-1}(X),$$

and thus, each  $U_{+1}(\mathbb{Z}_{2^n})$  transformation is exactly implementable by combining NOT, CNOT, and generalized Toffoli transformations only. Note that ancilla qubits are required only for realizing the generalized Toffoli transformations in this implementation. Accordingly, provided that generalized Toffoli transformations may be assumed available freely, the increment transformation over  $\mathbb{Z}_{2^n}$  may also be assumed available freely, for each positive integer  $n$ , and so may the controlled- $U_{+1}(\mathbb{Z}_{2^n})$  transformation  $\Lambda(U_{+1}(\mathbb{Z}_{2^n}))$ , by its construction.

**Threshold-check transformations** For any integers  $t$  and  $z$ , let  $f_{\geq t}: \mathbb{Z} \rightarrow \{0, 1\}$  denote the function defined by

$$f_{\geq t}(z) = \begin{cases} 0 & \text{if } z < t, \\ 1 & \text{if } z \geq t. \end{cases}$$

For any positive integer  $n$  and for an integer  $t$  in  $\mathbb{Z}_{2^n}$ , The threshold-check transformation  $U_{\geq t}(\mathbb{Z}_{2^n})$  over  $\mathbb{Z}_{2^n}$  is the unitary transformation acting over  $n+1$  qubits defined by

$$U_{\geq t}(\mathbb{Z}_{2^n}): |b\rangle \otimes |j\rangle \mapsto |b \oplus f_{\geq t}(j)\rangle \otimes |j\rangle, \quad \forall b \in \{0, 1\}, \forall j \in \mathbb{Z}_{2^n}.$$

The threshold-check transformation  $U_{\geq t}(\mathbb{Z}_{2^n})$  over  $\mathbb{Z}_{2^n}$  is easily implemented as follows, combining the increment transformations  $U_{+1}(\mathbb{Z}_{2^{n+1}})$  and  $U_{+1}(\mathbb{Z}_{2^n})$  over  $\mathbb{Z}_{2^{n+1}}$  and  $\mathbb{Z}_{2^n}$ , respectively:

$$U_{\geq t}(\mathbb{Z}_{2^n}) = (I \otimes U_{+1}(\mathbb{Z}_{2^n}))^t (U_{+1}(\mathbb{Z}_{2^{n+1}}))^{2^n-t}.$$

Accordingly, provided that generalized Toffoli transformations may be assumed available freely, the threshold-check transformation  $U_{\geq t}(\mathbb{Z}_{2^n})$  over  $\mathbb{Z}_{2^n}$  may also be assumed available freely, for each positive integer  $n$ .

### 3.4 $k$ -Clean-Qubit Computation and Complexity Classes

For any positive integer  $k$ , a *quantum computation with  $k$  clean qubits*, or simply a  *$k$ -clean-qubit computation*, is a computation performed by a unitary quantum circuit  $Q$  acting over  $w$  qubits, where  $w$  is a positive integer satisfying  $w \geq k$ . It is assumed that one of the qubits to which the circuit  $Q$  is applied is designated as the output qubit. The  $k$ -clean-qubit computation specified by the circuit  $Q$  proceeds as follows. For simplicity, we identify the quantum circuit  $Q$  with the unitary operator it induces. The initial state of the computation is the  $w$ -qubit state

$$\rho_{\text{init}}^{(w,k)} = (|0\rangle\langle 0|)^{\otimes k} \otimes \left(\frac{I}{2}\right)^{\otimes(w-k)}.$$

The circuit  $Q$  is applied to this initial state, which generates the  $w$ -qubit state

$$\rho_{\text{final}} = Q \rho_{\text{init}}^{(w,k)} Q^\dagger.$$

Now the designated output qubit is measured in the computational basis, where the outcome  $|0\rangle$  is interpreted as “accept”, and the outcome  $|1\rangle$  is interpreted as “reject”. Such a computation may also be called a *quantum computation of DQCk type*, or simply a *DQCk computation*, in analogy to the DQC1 computation.

The complexity classes  $Q_{[f]}P(c, s)$  and  $Q_fP(c, s)$  are defined as follows.

**Definition 11.** Given a function  $f: \mathbb{Z}^+ \rightarrow \mathbb{N}$  and functions  $c, s: \mathbb{Z}^+ \rightarrow [0, 1]$  satisfying  $c > s$ , a promise problem  $A = (A_{\text{yes}}, A_{\text{no}})$  is in  $\text{Q}_{[f]}P(c, s)$  iff there exists a polynomial-time uniformly generated family  $\{Q_x\}_{x \in \Sigma^*}$  of quantum circuits such that, for every  $x$  in  $\Sigma^*$ ,  $Q_x$  acts over  $w(|x|)$  qubits for some polynomially bounded function  $w: \mathbb{Z}^+ \rightarrow \mathbb{N}$  satisfying  $w \geq f$  and has the following properties:

- (**Completeness**) if  $x$  is in  $A_{\text{yes}}$ , the  $f(|x|)$ -clean-qubit computation induced by  $Q_x$  results in acceptance with probability at least  $c(|x|)$ ,
- (**Soundness**) if  $x$  is in  $A_{\text{no}}$ , the  $f(|x|)$ -clean-qubit computation induced by  $Q_x$  results in acceptance with probability at most  $s(|x|)$ .

**Definition 12.** Given a function  $f: \mathbb{Z}^+ \rightarrow \mathbb{N}$  and functions  $c, s: \mathbb{Z}^+ \rightarrow [0, 1]$  satisfying  $c > s$ , a promise problem  $A = (A_{\text{yes}}, A_{\text{no}})$  is in  $\text{Q}_fP(c, s)$  iff  $A$  is in  $\text{Q}_{[g]}P(c, s)$  for some function  $g: \mathbb{Z}^+ \rightarrow \mathbb{N}$  satisfying  $g \in O(f)$ .

Using these definitions, the complexity classes  $\text{BQ}_{[f]}P$  and  $\text{BQ}_fP$  are defined as follows.

**Definition 13.** Given a function  $f: \mathbb{Z}^+ \rightarrow \mathbb{N}$ , a promise problem  $A = (A_{\text{yes}}, A_{\text{no}})$  is in  $\text{BQ}_{[f]}P$  iff  $A$  is in  $\text{Q}_{[f]}P(\frac{2}{3}, \frac{1}{3})$ .

**Definition 14.** Given a function  $f: \mathbb{Z}^+ \rightarrow \mathbb{N}$ , a promise problem  $A = (A_{\text{yes}}, A_{\text{no}})$  is in  $\text{BQ}_fP$  iff  $A$  is in  $\text{Q}_fP(\frac{2}{3}, \frac{1}{3})$ .

Some remarks are in order on the definitions of  $\text{BQ}_{[f]}P$  and  $\text{BQ}_fP$ .

By Theorem 4 to be proved, for any logarithmically bounded function  $f$ , the above definition of  $\text{BQ}_fP$  is equivalent to a more conservative definition where the class consists of problems that are in  $\text{Q}_fP(1 - \varepsilon, \varepsilon)$  for some negligible function  $\varepsilon: \mathbb{Z}^+ \rightarrow [0, 1]$ . Similarly, for any logarithmically bounded function  $f \geq 2$ , the class  $\text{BQ}_{[f]}P$  above is equivalent to the class of problems that are in  $\text{Q}_{[f]}P(1 - \varepsilon, \varepsilon)$  for some negligible function  $\varepsilon: \mathbb{Z}^+ \rightarrow [0, 1]$ . Another candidate of definitions is to use  $\text{BQ}_{[f]}P'$  and  $\text{BQ}_fP'$ , which are the unions of  $\text{Q}_{[f]}P(c, s)$  and  $\text{Q}_fP(c, s)$ , respectively, over all functions  $c, s: \mathbb{Z}^+ \rightarrow [0, 1]$  satisfying  $c - s \geq \frac{1}{p}$  for some polynomially bounded function  $p: \mathbb{Z}^+ \rightarrow \mathbb{N}$ . As the computation error can be reduced by Theorem 4 only when there is a constant gap between completeness and soundness in the starting computation, it remains open whether these  $\text{BQ}_{[f]}P'$  and  $\text{BQ}_fP'$  are equal to  $\text{BQ}_{[f]}P$  and  $\text{BQ}_fP$  defined above.

Finally, the complexity classes  $\text{NQ}_{[1]}P$  and  $\text{SBQ}_{[1]}P$  are defined as follows, which are the one-clean-qubit-computation analogues of  $\text{NQP}$  and  $\text{SBQP}$ , respectively.

**Definition 15.** A promise problem  $A = (A_{\text{yes}}, A_{\text{no}})$  is in  $\text{NQ}_{[1]}P$  iff  $A$  is in  $\text{Q}_{[1]}P(c, 0)$  for some positive-valued function  $c: \mathbb{Z}^+ \rightarrow (0, 1]$ .

**Definition 16.** Given a function  $f: \mathbb{Z}^+ \rightarrow \mathbb{N}$ , a promise problem  $A = (A_{\text{yes}}, A_{\text{no}})$  is in  $\text{SBQ}_{[1]}P$  iff  $A$  is in  $\text{Q}_{[1]}P(2^{-p}, 2^{-p-1})$  for some polynomially bounded function  $p: \mathbb{Z}^+ \rightarrow \mathbb{N}$ .

*Remark.* As will be proved in Subsection 8.2, the class  $\text{SBQ}_{[1]}P$  has the following amplification property similar to  $\text{SBQP}$  and  $\text{SBP}$ : a problem  $A = (A_{\text{yes}}, A_{\text{no}})$  is in  $\text{SBQ}_{[1]}P$  iff for any polynomially bounded function  $p: \mathbb{Z}^+ \rightarrow \mathbb{N}$ , there exists a polynomially bounded function  $q: \mathbb{Z}^+ \rightarrow \mathbb{N}$  such that  $A$  is in  $\text{Q}_{[1]}P(2^{-q} \cdot (1 - 2^{-p}), 2^{-q} \cdot 2^{-p})$ .

### 3.5 Remarks on Uniformity of Quantum Circuits

The definition of polynomial-time uniformly generated family of quantum circuits in Subsection 3.3 allows the input  $x$  to be “hard-coded” in the generated circuit  $Q_x$ , i.e., each circuit  $Q_x$  is allowed to depend on the input  $x$  itself. As mentioned before, a more standard definition of the polynomial-time uniformity is the one in which each



circuit generated depends only on the input length  $|x|$ , and the input  $x$  is given to the circuit as a read-only input. It is stressed that all the results in this paper remain valid even when this more standard “non-hard-coded” definition is adopted for the polynomial-time uniformity of quantum circuits. It is further stressed that all the results in this paper still remain valid even when the complexity classes in Subsection 3.4 are defined with the *logarithmic-space* uniformly generated family of quantum circuits, by suitably replacing polynomial-time computable functions by logarithmic-space computable functions in the statements of Theorems 1 and 2 and Corollary 3, and by changing the definitions of polynomially and logarithmically bounded functions by using logarithmic-space deterministic Turing machines and logarithmic-space computable functions. In this case, the completeness results of Theorem 6 hold even under logarithmic-space many-one reduction.

One thing to be mentioned is, however, that some complexity classes defined in Subsection 3.4, such as  $Q_{\log}P(c, s)$ ,  $Q_{[1]}P(c, s)$ ,  $BQ_{\log}P$ , and  $BQ_{[1]}P$ , may depend on the definition of the uniformity of quantum circuits. Indeed, with the “hard-coded” definition of polynomial-time uniformity, the class  $P$  is trivially contained in each class defined in Subsection 3.4, whereas it becomes unclear whether  $P$  is included in the bounded-error classes such as  $BQ_{\log}P$  and  $BQ_{[1]}P$  when the classes are defined with the standard “non-hard-coded” definition of polynomial-time uniformity and with logarithmic-space uniformity. There even exists an oracle relative to which  $P$  is not included in  $BQ_{[1]}P$ , when  $BQ_{[1]}P$  is defined with logarithmic-space uniformity [She06, She09]. In this regard, when using the standard “non-hard-coded” definition of polynomial-time uniformity, polynomial-time many-one reduction may be too powerful for the models discussed in that the reduction itself already has computational power enough to solve any problem in  $P$ , while it is unclear whether the models for which the completeness results are discussed have such computational power, although Theorem 6 itself is mathematically valid even in this case.

In short, which uniformity of either “hard-coded” polynomial-time, or standard “non-hard-coded” polynomial-time, or logarithmic-space is used to define these complexity classes *does not* affect the properties proved in the present paper, but may affect how large these complexity classes themselves are.

## 4 Building Blocks

First, some notations are summarized that are used throughout this section.

Consider any quantum circuit  $Q$  acting over  $w$  qubits. For any positive integer  $k \leq w$ , let  $p_{\text{acc}}(Q, k)$  denote the acceptance probability of the  $k$ -clean-qubit computation induced by  $Q$ . More precisely, for any positive integer  $k \leq w$ , let  $\rho_{\text{init}}^{(w, k)}$  be the  $w$ -qubit initial state defined by

$$\rho_{\text{init}}^{(w, k)} = (|0\rangle\langle 0|)^{\otimes k} \otimes \left(\frac{I}{2}\right)^{\otimes (w-k)},$$

and let  $\Pi_{\text{acc}}$  be the projection operator defined by

$$\Pi_{\text{acc}} = |0\rangle\langle 0| \otimes I^{\otimes (w-1)}.$$

Now, for any positive integer  $k \leq w$ , the acceptance probability  $p_{\text{acc}}(Q, k)$  is defined by

$$p_{\text{acc}}(Q, k) = \text{tr } \Pi_{\text{acc}} Q \rho_{\text{init}}^{(w, k)} Q^\dagger.$$

### 4.1 Simulating $k$ -Clean-Qubit Computation by One-Clean-Qubit Computation

This subsection presents a procedure, called the ONE-CLEAN-QUBIT SIMULATION PROCEDURE, that constructs another quantum circuit  $R$  from  $Q$  such that the  $k$ -clean-qubit computation induced by  $Q$  can be simulated by the one-clean-qubit computation induced by  $R$ . More formally, given a description of a quantum circuit  $Q$  (which also specifies the number  $w$  of qubits  $Q$  acts over) and an integer  $k$  that specifies the number of clean qubits used in

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### ONE-CLEAN-QUBIT SIMULATION PROCEDURE

1. Prepare a single-qubit register  $O$  a  $k$ -qubit register  $Q$ , and a  $(w - k)$ -qubit register  $R$ , where the qubit in  $O$  is supposed to be initially in state  $|0\rangle$ , while all the qubits in  $Q$  and  $R$  are supposed to be initially in the totally mixed state  $I/2$ .  
Apply the  $k$ -controlled- $H$  transformation  $\Lambda^k(H)$  to  $(O, Q)$  using the qubit in  $O$  as the target, and then apply the NOT transformation  $X$  to each of the qubits in  $Q$  (applying these transformations has essentially the same effect as conditionally applying  $H$  to  $O$  if all the qubits in  $Q$  are in state  $|0\rangle$ ).
  2. Apply  $Q$  to  $(Q, R)$ .
  3. Apply the phase-flip (i.e., multiply the phase by  $-1$ ) if the content of  $(O, Q^{(1)})$  is 11, where  $Q^{(1)}$  denotes the single-qubit register consisting of the first qubit of  $Q$ .
  4. Apply  $Q^\dagger$  to  $(Q, R)$ .
  5. Apply  $X$  to each of the qubits in  $Q$ , and then apply  $\Lambda^k(H)$  to  $(O, Q)$  using the qubit in  $O$  as the target (applying these transformations has essentially the same effect as conditionally applying  $H$  to  $O$  if all the qubits in  $Q$  are in state  $|0\rangle$ ). Measure the qubit in  $O$  in the computational basis. Accept if this results in  $|0\rangle$ , and reject otherwise.
- 

Figure 4: The ONE-CLEAN-QUBIT SIMULATION PROCEDURE induced by a quantum circuit  $Q$  with the specification of the number  $k$  of clean qubits used in the computation of  $Q$  to be simulated.

the original computation to be simulated, the ONE-CLEAN-QUBIT SIMULATION PROCEDURE corresponds to a quantum circuit  $R$  such that the original  $k$ -clean-qubit computation induced by  $Q$  is simulated by the one-clean-qubit computation induced by  $R$ .

The circuit  $R$  acts over  $(w + 1)$  qubits, which are divided into three quantum registers: a single-qubit register  $O$ , a  $k$ -qubit register  $Q$ , and a  $(w - k)$ -qubit register  $R$ . It is supposed that the qubit in  $O$  is initially in state  $|0\rangle$ , and all the qubits in  $Q$  and  $R$  are initially in the totally mixed state  $I/2$ . Let  $Q^{(1)}$  denote the single-qubit register consisting of the first qubit of  $Q$ , which corresponds to the output qubit of  $Q$ . First, the circuit  $R$  applies the Hadamard transformation  $H$  to  $O$  if all the qubits in  $Q$  are in state  $|0\rangle$ . Next,  $R$  applies  $Q$  to  $(Q, R)$ , where the qubits in  $Q$  correspond to the  $k$  clean qubits of the original  $k$ -clean-qubit computation induced by  $Q$ . Now  $R$  flips the phase if and only if  $(O, Q^{(1)})$  contains 11 (i.e.,  $R$  applies the controlled- $Z$  transformation  $\Lambda(Z)$  to  $(O, Q^{(1)})$ ).  $R$  then applies  $Q^\dagger$  to  $(Q, R)$ , and further applies  $H$  to  $O$  if all the qubits in  $Q$  are in state  $|0\rangle$ . Finally, the qubit in  $O$  is measured in the computational basis, and  $R$  outputs the measurement result.

For the actual construction, the first conditional application of  $H$  to  $O$  when all the qubits in  $Q$  are in state  $|0\rangle$  is essentially realized by first applying the  $k$ -controlled- $H$  transformation  $\Lambda^k(H)$  to  $(O, Q)$  using the qubit in  $O$  as the target, and then applying the NOT transformation  $X$  to each of the qubits in  $Q$ . Similarly, the second conditional application of  $H$  to  $O$  is essentially realized by first applying  $X$  to each of the qubits in  $Q$ , and then applying  $\Lambda^k(H)$  to  $(O, Q)$  using the qubit in  $O$  as the target. Figures 4 and 5 summarize the construction of  $R$ .

**Proposition 17.** *For any quantum circuit  $Q$  and any positive integer  $k$ , let  $R$  be the quantum circuit corresponding to the ONE-CLEAN-QUBIT SIMULATION PROCEDURE induced by  $Q$  and  $k$ . For the acceptance probability  $p_{\text{acc}}(Q, k)$  of the  $k$ -clean-qubit computation induced by  $Q$  and the acceptance probability  $p_{\text{acc}}(R, 1)$  of the one-clean-qubit computation induced by  $R$ , it holds that*

$$1 - 2^{-k}(1 - (p_{\text{acc}}(Q, k))^2) \leq p_{\text{acc}}(R, 1) \leq 1 - 2^{-k}(1 - p_{\text{acc}}(Q, k)).$$

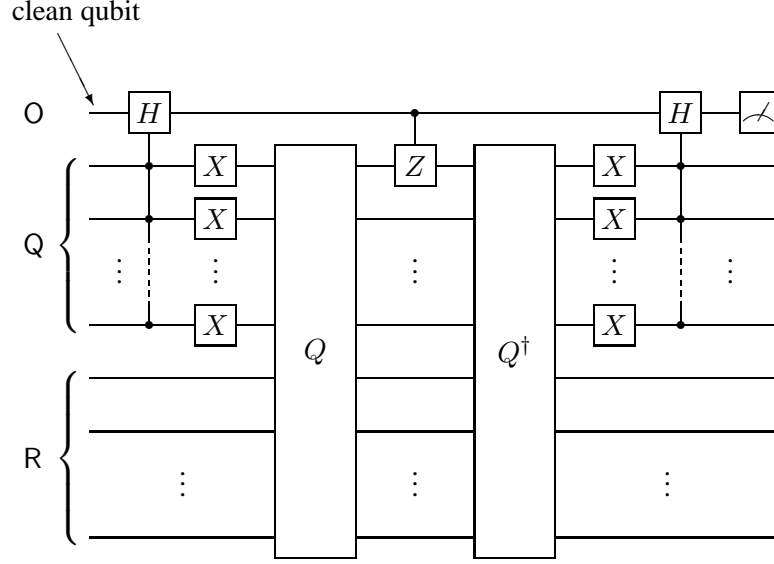


Figure 5: The quantum circuit for the ONE-CLEAN-QUBIT SIMULATION PROCEDURE induced by a quantum circuit  $Q$  with a specification of the quantum register  $Q$ , where the qubit in  $O$  is supposed to be the only clean qubit at the beginning of the computation, and is also the output qubit of the computation. The combinations of the controlled-Hadamard transformation and the NOT transformations  $X$  essentially correspond to applying the Hadamard transformation if all the qubits in  $Q$  are in state  $|0\rangle$ .

*Proof.* To analyze the acceptance probability  $p_{acc}(R, 1)$  of the one-clean-qubit computation induced by  $R$ , suppose that the content of  $Q$  is initially  $q$  in  $\Sigma^k$ , and the content of  $R$  is initially  $r$  in  $\Sigma^{(w-k)}$ .

First consider the case where  $q$  is not all-zero. In this case, nothing is applied to the qubit in  $O$  in Step 1, and thus the phase-flip is never performed in Step 3. Therefore, the application of  $Q^\dagger$  in Step 4 cancels out the application of  $Q$  in Step 2, and thus the content of  $Q$  remains  $q$  in Step 5, which is not all-zero. Hence, nothing is applied to the qubit in  $O$  in Step 5, either, and  $R$  outputs 0 and accepts.

Now consider the case where  $q$  is all-zero. In this case, by letting  $|\psi_r\rangle = |0\rangle^{\otimes k} \otimes |r\rangle$  the state in  $(O, Q, R)$  just after Step 2 is given by

$$\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes (Q|\psi_r\rangle).$$

For each  $j$  in  $\{0, 1\}$ , let  $\Pi_j$  be the projection operator acting over  $w$  qubits defined by  $\Pi_j = |j\rangle\langle j| \otimes I^{\otimes(w-1)}$  (notice that  $\Pi_0$  is nothing but  $\Pi_{acc}$ ). Then the state in  $(O, Q, R)$  just after Step 4 is given by

$$\begin{aligned} & \frac{1}{\sqrt{2}}|0\rangle \otimes |\psi_r\rangle + \frac{1}{\sqrt{2}}|1\rangle \otimes [Q^\dagger(\Pi_0 - \Pi_1)Q|\psi_r\rangle] \\ &= \frac{1}{\sqrt{2}}|0\rangle \otimes |\psi_r\rangle + \frac{1}{\sqrt{2}}|1\rangle \otimes [Q^\dagger(2\Pi_0 - I^{\otimes w})Q|\psi_r\rangle] \\ &= \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \otimes |\psi_r\rangle + \sqrt{2}|1\rangle \otimes (Q^\dagger\Pi_0 Q|\psi_r\rangle). \end{aligned}$$

Further define the projection operators  $\Delta_0$  and  $\Delta_1$  acting over  $w$  qubits by  $\Delta_0 = (|0\rangle\langle 0|)^{\otimes k} \otimes I^{\otimes(w-k)}$  and

$\Delta_1 = I^{\otimes w} - \Delta_0$ . Then the state in (O, Q, R) just after Step 5 is given by

$$|1\rangle \otimes |\psi_r\rangle + (|0\rangle - |1\rangle) \otimes (\Delta_0 Q^\dagger \Pi_0 Q |\psi_r\rangle) + \sqrt{2} |1\rangle \otimes (\Delta_1 Q^\dagger \Pi_0 Q |\psi_r\rangle).$$

Hence, the probability that  $R$  outputs 0 is given by  $\|\Delta_0 Q^\dagger \Pi_0 Q |\psi_r\rangle\|^2$ .

It follows that the overall acceptance probability  $p_{\text{acc}}(R, 1)$  of the one-clean-qubit computation induced by  $R$  is given by

$$p_{\text{acc}}(R, 1) = (1 - 2^{-k}) + 2^{-k} \cdot 2^{-(w-k)} \sum_{r \in \Sigma^{(w-k)}} \|\Delta_0 Q^\dagger \Pi_0 Q |\psi_r\rangle\|^2. \quad (1)$$

First notice that  $\Delta_0 = (|0\rangle\langle 0|)^{\otimes k} \otimes \sum_{r \in \Sigma^{(w-k)}} |r\rangle\langle r| = \sum_{r \in \Sigma^{(w-k)}} |\psi_r\rangle\langle \psi_r|$ , and thus, it holds that

$$\|\Delta_0 Q^\dagger \Pi_0 Q |\psi_r\rangle\| \geq \| |\psi_r\rangle\langle \psi_r| Q^\dagger \Pi_0 Q |\psi_r\rangle \| = |\langle \psi_r | Q^\dagger \Pi_0 Q |\psi_r\rangle| = \|\Pi_0 Q |\psi_r\rangle\|^2,$$

which is exactly the acceptance probability of the circuit  $Q$  when the input state to it was  $|\psi_r\rangle = |0\rangle^{\otimes k} \otimes |r\rangle$ . As the acceptance probability  $p_{\text{acc}}(Q, k)$  of the  $k$ -clean-qubit computation induced by  $Q$  is nothing but the expected value of  $\|\Pi_0 Q |\psi_r\rangle\|^2$  over  $r$  in  $\Sigma^{(w-k)}$ , it follows that

$$\begin{aligned} & 2^{-(w-k)} \sum_{r \in \Sigma^{(w-k)}} \|\Delta_0 Q^\dagger \Pi_0 Q |\psi_r\rangle\|^2 \\ & \geq \left( 2^{-(w-k)} \sum_{r \in \Sigma^{(w-k)}} \|\Delta_0 Q^\dagger \Pi_0 Q |\psi_r\rangle\| \right)^2 \geq \left( 2^{-(w-k)} \sum_{r \in \Sigma^{(w-k)}} \|\Pi_0 Q |\psi_r\rangle\|^2 \right)^2 = (p_{\text{acc}}(Q, k))^2. \end{aligned}$$

Combined with the equation (1), this implies that

$$p_{\text{acc}}(R, 1) \geq (1 - 2^{-k}) + 2^{-k} (p_{\text{acc}}(Q, k))^2 = 1 - 2^{-k} (1 - (p_{\text{acc}}(Q, k))^2),$$

which provides the first inequality.

Now notice that  $\|\Delta_0 Q^\dagger \Pi_0 Q |\psi_r\rangle\|^2$  is at most  $\|\Pi_0 Q |\psi_r\rangle\|^2$ , which is again exactly the acceptance probability of the circuit  $Q$  when the input state to it was  $|\psi_r\rangle = |0\rangle^{\otimes k} \otimes |r\rangle$ . Again using the fact that  $p_{\text{acc}}(Q, k)$  is nothing but the expected value of  $\|\Pi_0 Q |\psi_r\rangle\|^2$  over  $r$  in  $\Sigma^{(w-k)}$ , it holds that

$$2^{-(w-k)} \sum_{r \in \Sigma^{(w-k)}} \|\Delta_0 Q^\dagger \Pi_0 Q |\psi_r\rangle\|^2 \leq 2^{-(w-k)} \sum_{r \in \Sigma^{(w-k)}} \|\Pi_0 Q |\psi_r\rangle\|^2 = p_{\text{acc}}(Q, k).$$

Combined with the equation (1), this implies that

$$p_{\text{acc}}(R, 1) \leq (1 - 2^{-k}) + 2^{-k} p_{\text{acc}}(Q, k) = 1 - 2^{-k} (1 - p_{\text{acc}}(Q, k)),$$

and the second inequality follows.  $\square$

## 4.2 Amplifying Randomness of One-Clean-Qubit Computation

This subsection presents a procedure, called the RANDOMNESS AMPLIFICATION PROCEDURE, that constructs another quantum circuit  $R^{(N)}$  from  $Q$  when a positive integer  $N$  is specified. The circuit  $R^{(N)}$  is designed so that the sequence  $\{p_{\text{acc}}(R^{(N)}, 1)\}_{N \in \mathbb{N}}$  of the acceptance probability of the one-clean-qubit computation induced by  $R^{(N)}$  converges linearly to  $1/2$  with a rate related to the acceptance probability  $p_{\text{acc}}(Q, 1)$  of the one-clean-qubit computation induced by  $Q$ , if  $0 < p_{\text{acc}}(Q, 1) < 1$ .

For each  $N$  in  $\mathbb{N}$ , the circuit  $R^{(N)}$  acts over  $Nw + 1$  qubits, which are divided into  $2N + 1$  quantum registers: a single-qubit register O, and a  $(w - 1)$ -qubit register  $R_j$  and a single-qubit register  $X_j$  for each  $j$  in  $\{1, \dots, N\}$ .

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### RANDOMNESS AMPLIFICATION PROCEDURE

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1. Prepare a single-qubit register  $O$ , where the qubit in  $O$  is supposed to be initially in state  $|0\rangle$ . Prepare a  $(w-1)$ -qubit register  $R_j$  and a single-qubit register  $X_j$ , for each  $j$  in  $\{1, \dots, N\}$ , where all the qubits in  $R_j$  and  $X_j$  are supposed to be initially in the totally mixed state  $I/2$ .
  2. For  $j = 1$  to  $N$ , perform the following:
    - 2.1. Apply  $Q$  to  $(O, R_j)$ .
    - 2.2. Apply the CNOT transformation to  $(O, X_j)$  with the qubit in  $O$  being the control.
  3. Measure the qubit in  $O$  in the computational basis. Accept if this results in  $|0\rangle$ , and reject otherwise.
- 

Figure 6: The RANDOMNESS AMPLIFICATION PROCEDURE.

It is supposed that the qubit in  $O$  is initially in state  $|0\rangle$ , and all the qubits in  $R_j$  and  $X_j$  are initially in the totally mixed state  $I/2$ , for all  $j$  in  $\{1, \dots, N\}$ . For  $j = 1$  to  $N$ , the circuit  $R^{(N)}$  repeats the process of first applying  $Q$  to  $(O, R_j)$  and then applying the CNOT transformation to  $(O, X_j)$  using the qubit in  $O$  as the control, where each application of the CNOT transformation essentially has the same effect as measuring the qubit in  $O$  in the computational basis every time after  $Q$  is applied. Finally, the qubit in  $O$  is measured in the computational basis, and  $R^{(N)}$  outputs the measurement result. Figures 6 and 7 summarize the construction of  $R^{(N)}$ . (Strictly speaking, when  $j = N$  at Step 2 of Figure 6, it is redundant to apply CNOT at Step 2.2, as the qubit in  $O$  is anyway measured at Step 3 in the computational basis. This point is reflected in Figure 7.)

**Proposition 18.** *For any quantum circuit  $Q$  and any positive integer  $N$ , let  $R^{(N)}$  be the quantum circuit corresponding to the RANDOMNESS AMPLIFICATION PROCEDURE induced by  $Q$  and  $N$ . For the acceptance probability  $p_{\text{acc}}(Q, 1)$  of the one-clean-qubit computation induced by  $Q$  and the acceptance probability  $p_{\text{acc}}(R^{(N)}, 1)$  of the one-clean-qubit computation induced by  $R^{(N)}$ , it holds that*

$$p_{\text{acc}}(R^{(N)}, 1) = \frac{1}{2} + \frac{1}{2}(2p_{\text{acc}}(Q, 1) - 1)^N.$$

*Proof.* For each  $j$  in  $\{0, 1\}$ , let  $\Pi_j$  be the projection operator acting over  $w$  qubits defined by  $\Pi_j = |j\rangle\langle j| \otimes I^{\otimes(w-1)}$ , and let  $\rho_j$  be the quantum state of  $w$  qubits defined by  $\rho_j = |j\rangle\langle j| \otimes (\frac{I}{2})^{\otimes(w-1)}$ . Note that  $\Pi_0 = \Pi_{\text{acc}}$  and  $\rho_0 = \rho_{\text{init}}^{(w,1)}$ , and thus, the acceptance probability  $p_{\text{acc}}(Q, 1)$  of the original one-clean-qubit computation induced by  $Q$  is given by

$$p_{\text{acc}}(Q, 1) = \text{tr } \Pi_0 Q \rho_0 Q^\dagger.$$

By noticing that  $\Pi_1 = I^{\otimes w} - \Pi_0$  and

$$\text{tr } \Pi_0 Q \rho_0 Q^\dagger + \text{tr } \Pi_0 Q \rho_1 Q^\dagger = \frac{1}{2^{w-1}} \text{tr } \Pi_0 Q (I^{\otimes w}) Q^\dagger = \frac{1}{2^{w-1}} \text{tr } \Pi_0 = 1,$$

it also holds that

$$\text{tr } \Pi_1 Q \rho_1 Q^\dagger = 1 - \text{tr } \Pi_0 Q \rho_1 Q^\dagger = 1 - (1 - \text{tr } \Pi_0 Q \rho_0 Q^\dagger) = \text{tr } \Pi_0 Q \rho_0 Q^\dagger = p_{\text{acc}}(Q, 1).$$

This implies that, after each repetition of Step 2 of the RANDOMNESS AMPLIFICATION PROCEDURE, the content of  $O$  remains unchanged with probability  $p_{\text{acc}}(Q, 1)$ , and is flipped with probability  $1 - p_{\text{acc}}(Q, 1)$  (by viewing

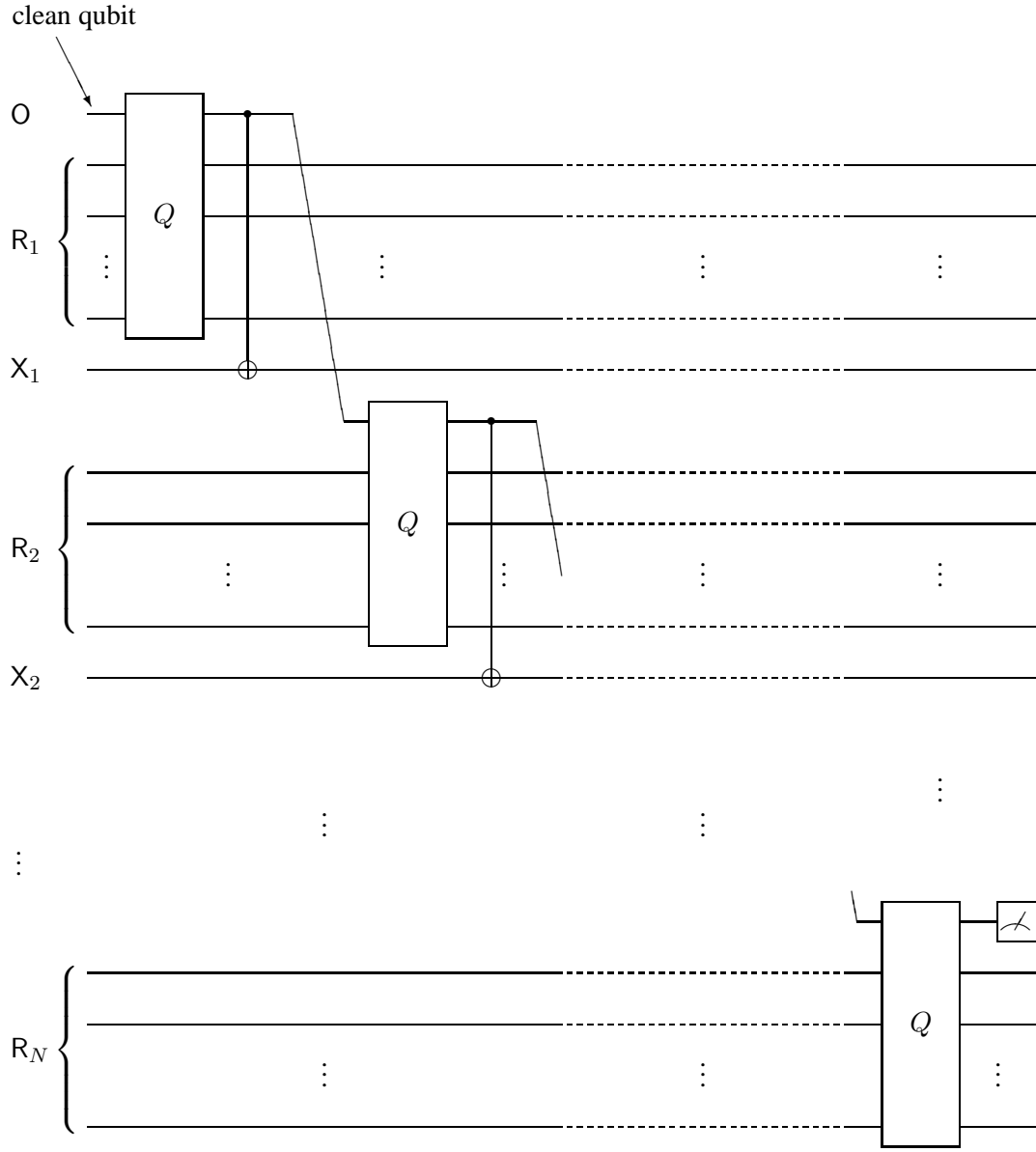


Figure 7: The quantum circuit for the RANDOMNESS AMPLIFICATION PROCEDURE induced by a quantum circuit  $Q$ , where the qubit in  $O$  is supposed to be the only clean qubit at the beginning of the computation, and is also the output qubit of the computation.

that  $O$  is “measured” in the computational basis as a result of the application of CNOT at Step 2.2). The content of  $O$  is 0 when entering Step 3 if and only if the content of  $O$  is flipped even number of times during Step 2. By Lemma 10, this happens with probability exactly  $\frac{1}{2} + \frac{1}{2}(2p_{\text{acc}}(Q, 1) - 1)^N$ , which gives the acceptance probability  $p_{\text{acc}}(R^{(N)}, 1)$  of the one-clean-qubit computation induced by  $R^{(N)}$ .  $\square$

### 4.3 Checking Stability of One-Clean-Qubit Computation

This subsection considers two procedures of checking whether a given one-clean-qubit computation has sufficiently high acceptance probability or not. For a positive integer  $N$  specified, the first procedure, called the ONE-CLEAN-QUBIT STABILITY CHECKING PROCEDURE, constructs another quantum circuit  $R_1^{(N)}$  from  $Q$  so that the acceptance probability  $p_{\text{acc}}(R_1^{(N)}, 1)$  of the one-clean-qubit computation induced by  $R_1^{(N)}$  is polynomially small with respect to  $N$  when the acceptance probability was sufficiently close to  $1/2$  in the original one-clean-qubit computation induced by  $Q$ . The second procedure, called the TWO-CLEAN-QUBIT STABILITY CHECKING PROCEDURE, is a modification of the first procedure so that, using two clean qubits, the acceptance probability  $p_{\text{acc}}(R_2^{(N)}, 2)$  of the two-clean-qubit computation induced by the circuit  $R_2^{(N)}$  constructed from  $Q$  now becomes exponentially small with respect to  $N$  when the acceptance probability was sufficiently close to  $1/2$  in the original one-clean-qubit computation induced by  $Q$ .

#### 4.3.1 Stability Checking Using One Clean Qubit

First consider a slightly simplified version of the ONE-CLEAN-QUBIT STABILITY CHECKING PROCEDURE described below.

For each  $N$  in  $\mathbb{N}$  satisfying  $N \geq 2$ , the procedure to be constructed prepares  $2N + 1$  quantum registers: a single-qubit register  $Q$ , and a  $(w - 1)$ -qubit register  $R_j$  for each  $j$  in  $\{1, \dots, 2N\}$ . It is supposed that all the qubits in  $Q$  and  $R_j$  are initially in the totally mixed state  $I/2$ , for all  $j$  in  $\{1, \dots, 2N\}$ . The procedure also prepares an integer variable  $C$  that serves as a counter, but the value of  $C$  is not necessarily initialized to 0 at the beginning of the computation, and the initial value  $r$  of  $C$  is chosen from the set  $\{0, \dots, N - 1\}$  uniformly at random. For  $j = 1$  to  $2N$ , the procedure repeats the process of first applying  $Q$  to  $(Q, R_j)$  and then measuring the qubit in  $Q$  in the computational basis. Everytime this measurement results in  $|1\rangle$ , the value of the counter  $C$  is increased by one. After this repetition of  $2N$  times, the procedure checks the value of  $C$ , and rejects if it is between  $N$  and  $2N - 1$ .

For the actual construction of the above simplified procedure, the positive integer  $N$  is chosen to be a power of two so that the most significant bit is 0 for any integer in the interval  $[0, N - 1]$  and is 1 for any integer in the interval  $[N, 2N - 1]$ , when expressed as a binary string using  $\log N + 1$  bits. The actual procedure also introduces a  $(\log N + 1)$ -qubit register  $C$  and a single-qubit register  $X_j$  for each  $j$  in  $\{1, \dots, 2N\}$ . Only the first qubit of  $C$  is supposed to be initially in state  $|0\rangle$ , and all the other qubits used in the actual procedure are supposed to be initially in the totally mixed state  $I/2$ . The content of  $C$  serves as a counter  $C$  of the simplified procedure. The condition that only the first qubit in  $C$  is initially in state  $|0\rangle$  and all the other qubits in  $C$  are initially in the totally mixed state  $I/2$  exactly corresponds to the process in the simplified procedure of randomly picking the initial value of the counter  $C$  from the set  $\{0, \dots, N - 1\}$ . The conditional increment of the counter value is realized by the controlled- $U_{+1}(\mathbb{Z}_{2N})$  transformation  $\Lambda(U_{+1}(\mathbb{Z}_{2N}))$ . The final decision of acceptance and rejection of the simplified procedure can be done by measuring the first qubit of  $C$  in the computational basis. Each register  $X_j$  is used to simulate the measurement at each repetition round  $j$ . More precisely, for each repetition round  $j$ , the measurement of the qubit in  $Q$  in the computational basis is replaced by the application of a CNOT transformation to  $(Q, X_j)$ . Figures 8 and 9 summarize the actual construction of the ONE-CLEAN-QUBIT STABILITY CHECKING PROCEDURE.

First analyze the lower bound of the acceptance probability of the one-clean-qubit computation induced by the quantum circuit resulting from the ONE-CLEAN-QUBIT STABILITY CHECKING PROCEDURE.

**Proposition 19.** *For any quantum circuit  $Q$  and any positive integer  $N$  that is a power of two, let  $R_1^{(N)}$  be the quantum circuit corresponding to the ONE-CLEAN-QUBIT STABILITY CHECKING PROCEDURE induced by  $Q$  and  $N$ . For the acceptance probability  $p_{\text{acc}}(Q, 1)$  of the one-clean-qubit computation induced by  $Q$  and the acceptance probability  $p_{\text{acc}}(R_1^{(N)}, 1)$  of the one-clean-qubit computation induced by  $R_1^{(N)}$ , it holds that*

$$p_{\text{acc}}(R_1^{(N)}, 1) \geq (p_{\text{acc}}(Q, 1))^{2N-1}.$$

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### ONE-CLEAN-QUBIT STABILITY CHECKING PROCEDURE

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1. Given a positive integer  $N$  that is a power of two, let  $l = \log N + 1$ . Prepare an  $l$ -qubit register  $C$ , a single-qubit register  $Q$ , and each  $(w - 1)$ -qubit register  $R_j$  for each  $j$  in  $\{1, \dots, 2N\}$ . For each  $j$  in  $\{1, \dots, l\}$ , let  $C^{(j)}$  denote the single-qubit register corresponding to the  $j$ th qubit of  $C$ . The qubit in  $C^{(1)}$  is supposed to be initially in state  $|0\rangle$ , while the qubit in  $C^{(j)}$  for each  $j$  in  $\{2, \dots, l\}$ , the qubit in  $Q$ , and all the qubits in  $R_{j'}$  for each  $j'$  in  $\{1, \dots, 2N\}$  are supposed to be initially in the totally mixed state  $I/2$ . Prepare a single-qubit register  $X_j$  for each  $j$  in  $\{1, \dots, 2N\}$ , where the qubit in each  $X_j$  is supposed to be initially in the totally mixed state  $I/2$ .
  2. For  $j = 1$  to  $2N$ , perform the following:
    - 2.1. Apply  $Q$  to  $(Q, R_j)$ .
    - 2.2. Apply the CNOT transformation to  $(Q, X_j)$  with the qubit in  $Q$  being the control. Apply the controlled- $U_{+1}(\mathbb{Z}_{2N})$  transformation  $\Lambda(U_{+1}(\mathbb{Z}_{2N}))$  to  $(Q, C)$  with the qubit in  $Q$  being the control.
  3. Measure the qubit in  $C^{(1)}$  in the computational basis. Accept if this results in  $|0\rangle$ , and reject otherwise.
- 

Figure 8: The ONE-CLEAN-QUBIT STABILITY CHECKING PROCEDURE.

*Proof.* As in Subsection 4.2, for each  $j$  in  $\{0, 1\}$ , let  $\Pi_j$  be the projection operator acting over  $w$  qubits defined by  $\Pi_j = |j\rangle\langle j| \otimes I^{\otimes(w-1)}$ , and let  $\rho_j$  be the quantum state of  $w$  qubits defined by  $\rho_j = |j\rangle\langle j| \otimes \left(\frac{I}{2}\right)^{\otimes(w-1)}$ . The acceptance probability  $p_{\text{acc}}(Q, 1)$  of the original one-clean-qubit computation induced by  $Q$  is given by

$$p_{\text{acc}}(Q, 1) = \text{tr } \Pi_0 Q \rho_0 Q^\dagger,$$

and it holds that

$$\text{tr } \Pi_1 Q \rho_1 Q^\dagger = p_{\text{acc}}(Q, 1).$$

This implies that, for each repetition round during Step 2, the counter value stored in  $C$  is increased by one with probability  $1 - p_{\text{acc}}(Q, 1)$  if the content of  $Q$  was 0 when entering Step 2.1, while it is increased by one with probability  $p_{\text{acc}}(Q, 1)$  if the content of  $Q$  was 1 when entering Step 2.1.

Notice that, at the  $j$ th repetition round in Step 2 for  $j \geq 2$ , the content of  $Q$  is 1 when entering Step 2.1 if and only if the previous repetition round has increased the counter value in  $C$ . Hence, taking into account that the content of  $Q$  is initially 0 or 1 with equal probability, after all the repetition rounds of Step 2, the counter value in  $C$  is never increased with probability

$$\frac{1}{2}(p_{\text{acc}}(Q, 1))^{2N} + \frac{1}{2}(1 - p_{\text{acc}}(Q, 1))(p_{\text{acc}}(Q, 1))^{2N-1} = \frac{1}{2}(p_{\text{acc}}(Q, 1))^{2N-1},$$

while it is increased by  $2N$  with probability

$$\frac{1}{2}(1 - p_{\text{acc}}(Q, 1))(p_{\text{acc}}(Q, 1))^{2N-1} + \frac{1}{2}(p_{\text{acc}}(Q, 1))^{2N} = \frac{1}{2}(p_{\text{acc}}(Q, 1))^{2N-1}.$$

The content of  $C$  stationarily remains its initial value  $r \leq N - 1$  for the first case, and it comes back to the initial value  $r \leq N - 1$  for the second case. As Step 3 results in acceptance at least in these two cases, it follows that

$$p_{\text{acc}}(R_1^{(N)}, 1) \geq \frac{1}{2}(p_{\text{acc}}(Q, 1))^{2N-1} + \frac{1}{2}(p_{\text{acc}}(Q, 1))^{2N-1} = (p_{\text{acc}}(Q, 1))^{2N-1},$$

as claimed. □



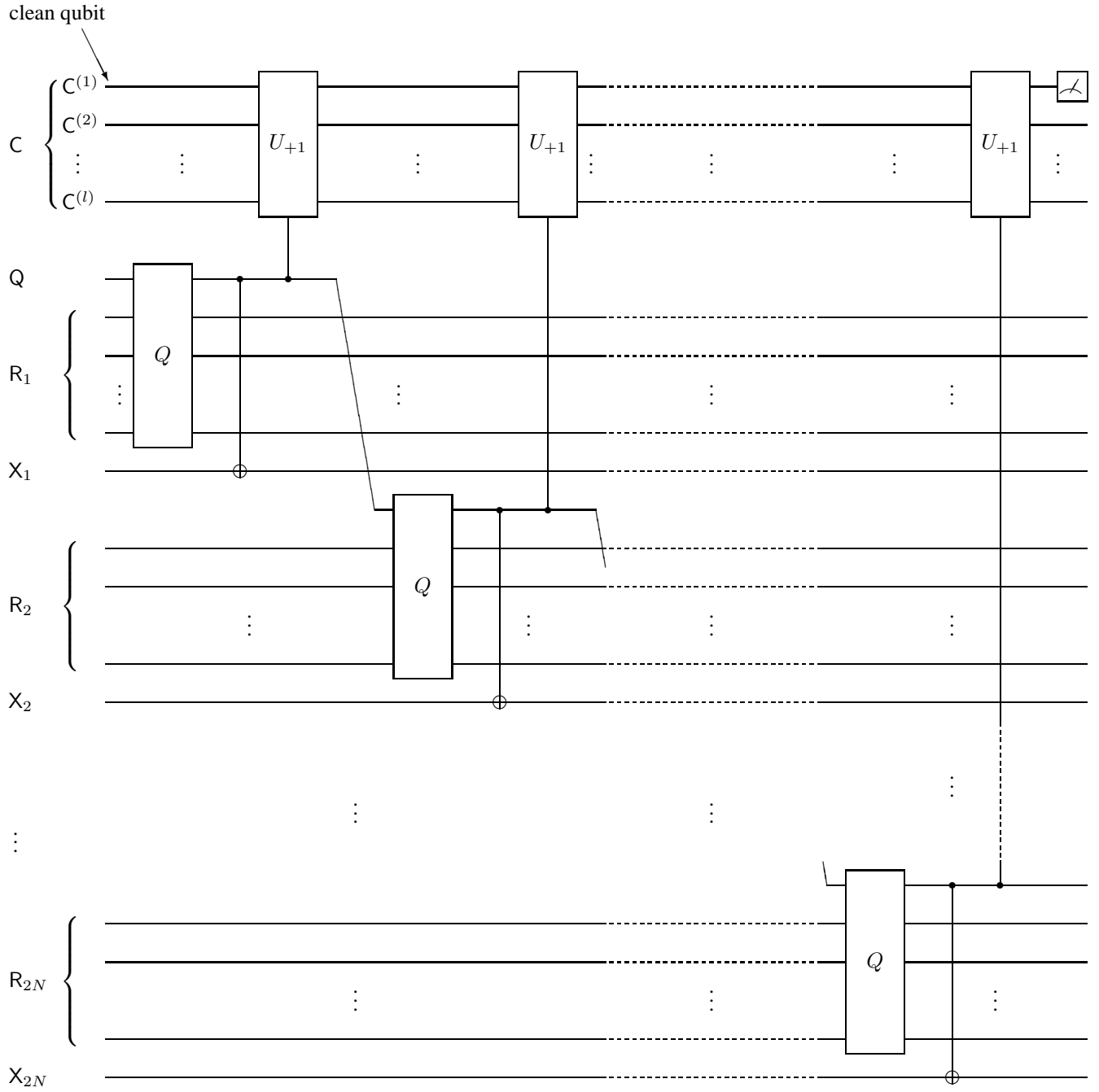


Figure 9: The quantum circuit for the ONE-CLEAN-QUBIT STABILITY CHECKING PROCEDURE induced by a quantum circuit  $Q$  and a positive integer  $N$ , where  $U_{+1}$  is a shorthand of the increment transformation  $U_{+1}(\mathbb{Z}_{2N})$ . The qubit in  $C^{(1)}$  is supposed to be the only clean qubit at the beginning of the computation, and is also the output qubit of the computation.

Proposition 19 in particular implies that, if the original acceptance probability  $p_{\text{acc}}(Q, 1)$  is one, the acceptance probability  $p_{\text{acc}}(R_1^{(N)}, 1)$  is also one for the circuit  $R_1^{(N)}$  corresponding to the ONE-CLEAN-QUBIT STABILITY CHECKING PROCEDURE induced by  $Q$  and  $N$ .

The next proposition provides an upper bound of the acceptance probability of the one-clean-qubit computation induced by the quantum circuit resulting from the ONE-CLEAN-QUBIT STABILITY CHECKING PROCEDURE, assuming that the original acceptance probability  $p_{\text{acc}}(Q, 1)$  is close to  $1/2$ .

**Proposition 20.** *For any quantum circuit  $Q$  and any positive integer  $N$  that is a power of two and at least  $2^6 = 64$ , let  $R_1^{(N)}$  be the quantum circuit corresponding to the ONE-CLEAN-QUBIT STABILITY CHECKING PROCEDURE induced by  $Q$  and  $N$ . If the acceptance probability  $p_{\text{acc}}(Q, 1)$  of the one-clean-qubit computation induced by  $Q$  satisfies that  $\frac{1}{2} - \varepsilon \leq p_{\text{acc}}(Q, 1) \leq \frac{1}{2} + \varepsilon$  for some  $\varepsilon \in [0, \frac{1}{8}]$  such that  $3N^{-\frac{1}{3}} + 4\varepsilon \leq 1$ , it holds for the acceptance probability  $p_{\text{acc}}(R_1^{(N)}, 1)$  of the one-clean-qubit computation induced by  $R_1^{(N)}$  that*

$$p_{\text{acc}}(R_1^{(N)}, 1) < 3N^{-\frac{1}{3}} + 4\varepsilon.$$

*Proof.* As before, for each  $j$  in  $\{0, 1\}$ , let  $\Pi_j$  be the projection operator acting over  $w$  qubits defined by  $\Pi_j = |j\rangle\langle j| \otimes I^{\otimes(w-1)}$ , and let  $\rho_j$  be the quantum state of  $w$  qubits defined by  $\rho_j = |j\rangle\langle j| \otimes (\frac{I}{2})^{\otimes(w-1)}$ . The acceptance probability  $p_{\text{acc}}(Q, 1)$  of the original one-clean-qubit computation induced by  $Q$  is given by

$$p_{\text{acc}}(Q, 1) = \text{tr } \Pi_0 Q \rho_0 Q^\dagger,$$

and it holds that

$$\text{tr } \Pi_1 Q \rho_1 Q^\dagger = p_{\text{acc}}(Q, 1).$$

Notice that, for each repetition round during Step 2, the counter value in  $C$  is increased by one with probability at least  $\min\{p_{\text{acc}}(Q, 1), 1 - p_{\text{acc}}(Q, 1)\} \geq \frac{1}{2} - \varepsilon$  and at most  $\max\{p_{\text{acc}}(Q, 1), 1 - p_{\text{acc}}(Q, 1)\} \leq \frac{1}{2} + \varepsilon$  regardless of the content of  $Q$  being 0 or 1 when entering Step 2.1. Hence, from the Hoeffding bound (Lemma 9), the probability that the total increment of the counter value in  $C$  is at most  $(1 - 2\varepsilon - 2\delta)N$  after all the  $2N$  repetition rounds of Step 2 is less than  $e^{-4\delta^2 N}$ , for any  $\delta$  in  $[0, \frac{1}{2} - \varepsilon]$ . Similarly, the probability that the total increment of the counter value in  $C$  is at least  $(1 + 2\varepsilon + 2\delta)N$  after all the  $2N$  repetition rounds of Step 2 is less than  $e^{-4\delta^2 N}$  also, for any  $\delta$  in  $[0, \frac{1}{2} - \varepsilon]$ . It follows that, when  $\delta$  is in  $(\frac{1}{2\sqrt{N}}, \frac{1}{4} - \varepsilon + \frac{1}{4N}]$  and the initial counter value  $r$  satisfies that

$$2(\varepsilon + \delta)N - 1 \leq r \leq N - 2(\varepsilon + \delta)N,$$

after all the  $2N$  repetition rounds of Step 2, the probability that the counter value in  $C$  is in the interval  $[N, 2N - 1]$  is more than  $1 - 2e^{-4\delta^2 N} > 1 - 2^{-4\delta^2 N+1}$ , and thus, the acceptance probability at Step 3 is less than

$$\left[4(\varepsilon + \delta) - \frac{2}{N}\right] \cdot 1 + \left[1 - 4(\varepsilon + \delta) + \frac{2}{N}\right] \cdot 2^{-4\delta^2 N+1} < 4(\varepsilon + \delta) + 2^{-4\delta^2 N+1}.$$

By taking  $\delta$  to be  $\frac{1}{2}N^{-\frac{1}{3}}$  (which is in  $(\frac{1}{2\sqrt{N}}, \frac{1}{4} - \varepsilon + \frac{1}{4N}]$  as  $\varepsilon$  is at most  $1/8$  and  $N$  is at least  $2^6 = 64$ ), and using the fact that  $x \leq 2^{x-1}$  holds for any  $x \geq 2$ , it follows that

$$p_{\text{acc}}(R_1^{(N)}, 1) < 2N^{-\frac{1}{3}} + 2^{-N^{\frac{1}{3}}+1} + 4\varepsilon \leq 3N^{-\frac{1}{3}} + 4\varepsilon,$$

which completes the proof.  $\square$

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### TWO-CLEAN-QUBIT STABILITY CHECKING PROCEDURE

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1. Given a positive integer  $N$  that is a power of two, let  $l = \log N + 3$ . Prepare an  $l$ -qubit register  $C$ , a single-qubit register  $Q$ , and each  $(w - 1)$ -qubit register  $R_j$  for each  $j$  in  $\{1, \dots, 8N\}$ . For each  $j$  in  $\{1, \dots, l\}$ , let  $C^{(j)}$  denote the single-qubit quantum register corresponding to the  $j$ th qubit of  $C$ . The qubits in  $C^{(1)}$  and  $C^{(2)}$  are supposed to be initially in state  $|0\rangle$ , while the qubit in  $C^{(j)}$  for each  $j$  in  $\{3, \dots, l\}$ , the qubit in  $Q$ , and all the qubits in  $R_{j'}$  for each  $j'$  in  $\{1, \dots, 8N\}$  are supposed to be initially in the totally mixed state  $I/2$ . Prepare a single-qubit register  $X_j$  for each  $j$  in  $\{1, \dots, 8N\}$ , where the qubit in each  $X_j$  is supposed to be initially in the totally mixed state  $I/2$ .
  2. For  $j = 1$  to  $N$ , apply the unitary transformation  $U_{+1}(\mathbb{Z}_{8N})$  to  $C$ .
  3. For  $j = 1$  to  $8N$ , perform the following:
    - 3.1. Apply  $Q$  to  $(Q, R_j)$ .
    - 3.2. Apply the CNOT transformation to  $(Q, X_j)$  with the qubit in  $Q$  being the control. Apply the controlled- $U_{+1}(\mathbb{Z}_{8N})$  transformation  $\Lambda(U_{+1}(\mathbb{Z}_{8N}))$  to  $(Q, C)$  with the qubit in  $Q$  being the control.
  4. Measure the qubit in  $C^{(1)}$  in the computational basis. Accept if this results in  $|0\rangle$ , and reject otherwise.
- 

Figure 10: The TWO-CLEAN-QUBIT STABILITY CHECKING PROCEDURE.

#### 4.3.2 Stability Checking Using Two Clean Qubits

As can be seen in the proof of Proposition 20, one drawback of the ONE-CLEAN-QUBIT STABILITY CHECKING PROCEDURE when analyzing the upper bound of its acceptance probability is that there are some “bad” initial counter values with which the procedure is forced to accept with unallowably high probability even if the acceptance probability  $p_{\text{acc}}(Q, 1)$  of the underlying circuit  $Q$  was sufficiently close to  $1/2$ . This is the essential reason why only a polynomially small upper bound can be proved on the acceptance probability of the ONE-CLEAN-QUBIT STABILITY CHECKING PROCEDURE in Proposition 20. As there are only polynomially many number of possible initial counter values, even just one “bad” initial value is unacceptable to go beyond polynomially small upper bounds. The authors do not know how to get rid of this barrier with the use of just one clean qubit.

In contrast, if two clean qubits are available, one can easily modify the procedure so that it has no “bad” initial counter value. The idea is very simple. This time, the register  $C$  uses  $\log N + 3$  qubits so that the counter takes values in  $\mathbb{Z}_{8N} = \{0, \dots, 8N - 1\}$ . The first two qubits of  $C$  are supposed to be initially in state  $|0\rangle$ , which implies that the initial counter value is picked uniformly from the set  $\{0, \dots, 2N - 1\}$ . The point is that one can increase the counter value by  $N$  before the repetition starts so that the actual initial value of the counter is in the set  $\{N, \dots, 3N - 1\}$ . If the acceptance probability  $p_{\text{acc}}(Q, 1)$  of the underlying circuit  $Q$  was in the interval  $[\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon]$  for some sufficiently small  $\varepsilon$ , the expected value of the counter after the repetition must be in the interval  $[(5 - 8\varepsilon)N, (7 + 8\varepsilon)N - 1]$ . Hence, if  $\varepsilon$  is in  $[0, \frac{1}{16}]$ , for instance, the probability is exponentially small for the event that the final counter value is not in the interval  $[4N, 8N - 1]$ . This leads to the TWO-CLEAN-QUBIT STABILITY CHECKING PROCEDURE, whose construction is summarized in Figure 10.

First, the following lower bound holds for the acceptance probability of the two-clean-qubit computation induced by the quantum circuit resulting from the TWO-CLEAN-QUBIT STABILITY CHECKING PROCEDURE. In particular, if the original acceptance probability  $p_{\text{acc}}(Q, 1)$  is one, the acceptance probability  $p_{\text{acc}}(R_2^{(N)}, 2)$  is also one for the circuit  $R_2^{(N)}$  corresponding to the TWO-CLEAN-QUBIT STABILITY CHECKING PROCEDURE induced by  $Q$  and  $N$ .

**Proposition 21.** For any quantum circuit  $Q$  and any positive integer  $N$  that is a power of two, let  $R_2^{(N)}$  be the quantum circuit corresponding to the TWO-CLEAN-QUBIT STABILITY CHECKING PROCEDURE induced by  $Q$  and  $N$ . For the acceptance probability  $p_{\text{acc}}(Q, 1)$  of the one-clean-qubit computation induced by  $Q$  and the acceptance probability  $p_{\text{acc}}(R_2^{(N)}, 2)$  of the two-clean-qubit computation induced by  $R_2^{(N)}$ , it holds that

$$p_{\text{acc}}(R_2^{(N)}, 2) \geq (p_{\text{acc}}(Q, 1))^{8N-1}.$$

The proof of Proposition 21 is essentially the same as that of Proposition 19, and is omitted.

The next proposition provides an upper bound of the acceptance probability of the two-clean-qubit computation induced by the quantum circuit resulting from the TWO-CLEAN-QUBIT STABILITY CHECKING PROCEDURE, assuming that the original acceptance probability  $p_{\text{acc}}(Q, 1)$  is close to  $1/2$ .

**Proposition 22.** For any quantum circuit  $Q$  and any positive integer  $N$  that is a power of two and at least  $2^4 = 16$ , let  $R_2^{(N)}$  be the quantum circuit corresponding to the TWO-CLEAN-QUBIT STABILITY CHECKING PROCEDURE induced by  $Q$  and  $N$ . If the acceptance probability  $p_{\text{acc}}(Q, 1)$  of the one-clean-qubit computation induced by  $Q$  satisfies that  $\frac{1}{2} - \varepsilon \leq p_{\text{acc}}(Q, 1) \leq \frac{1}{2} + \varepsilon$  for some  $\varepsilon$  in  $[0, \frac{1}{16}]$ , it holds for the acceptance probability  $p_{\text{acc}}(R_2^{(N)}, 2)$  of the two-clean-qubit computation induced by  $R_2^{(N)}$  that

$$p_{\text{acc}}(R_2^{(N)}, 2) < 2^{-\frac{N}{16}+1}.$$

*Proof.* The proof is similar to that of Proposition 20. As before, for each  $j$  in  $\{0, 1\}$ , let  $\Pi_j$  be the projection operator acting over  $w$  qubits defined by  $\Pi_j = |j\rangle\langle j| \otimes I^{\otimes(w-1)}$ , and let  $\rho_j$  be the quantum state of  $w$  qubits defined by  $\rho_j = |j\rangle\langle j| \otimes (\frac{I}{2})^{\otimes(w-1)}$ . The acceptance probability  $p_{\text{acc}}(Q, 1)$  of the original one-clean-qubit computation induced by  $Q$  is given by

$$p_{\text{acc}}(Q, 1) = \text{tr } \Pi_0 Q \rho_0 Q^\dagger,$$

and it holds that

$$\text{tr } \Pi_1 Q \rho_1 Q^\dagger = p_{\text{acc}}(Q, 1).$$

Notice that, for each repetition round during Step 3, the counter value in  $C$  is increased by one with probability at least  $\min\{p_{\text{acc}}(Q, 1), 1 - p_{\text{acc}}(Q, 1)\} \geq \frac{1}{2} - \varepsilon$  and at most  $\max\{p_{\text{acc}}(Q, 1), 1 - p_{\text{acc}}(Q, 1)\} \leq \frac{1}{2} + \varepsilon$  regardless of the content of  $Q$  being 0 or 1 when entering Step 3.1. Hence, from the Hoeffding bound (Lemma 9), the probability that the total increment of the counter value in  $C$  is at most  $(4 - 8\varepsilon - 8\delta)N$  after all the  $8N$  repetition rounds of Step 3 is less than  $e^{-16\delta^2 N}$ , for any  $\delta$  in  $[0, \frac{1}{2} - \varepsilon]$ . Similarly, the probability that the total increment of the counter value in  $C$  is at least  $(4 + 8\varepsilon + 8\delta)N$  after all the  $8N$  repetition rounds of Step 3 is less than  $e^{-16\delta^2 N}$  also, for any  $\delta$  in  $[0, \frac{1}{2} - \varepsilon]$ . As the counter value  $r$  in  $C$  at the beginning of Step 3 satisfies that

$$N \leq r \leq 3N - 1,$$

when  $\delta$  is in  $(\frac{1}{4\sqrt{N}}, \frac{1}{8} - \varepsilon + \frac{1}{8N}]$ , it holds that, after all the  $8N$  repetition rounds of Step 3, the probability that the counter value in  $C$  is in the interval  $[4N, 8N - 1]$  is more than  $1 - 2e^{-16\delta^2 N} > 1 - 2^{-16\delta^2 N+1}$ , and thus, the acceptance probability at Step 4 is less than

$$1 - (1 - 2^{-16\delta^2 N+1}) = 2^{-16\delta^2 N+1}.$$

By taking  $\delta$  to be  $\frac{1}{16}$ , it follows that

$$p_{\text{acc}}(R_2^{(N)}, 2) < 2^{-\frac{N}{16}+1},$$

which completes the proof.  $\square$

## 5 Error Reduction for One-Sided-Error Cases

This section proves the error reduction results in the cases of one-sided bounded error. First, Subsection 5.1 treats the cases with perfect completeness, namely, Theorems 1 and 2. The cases with perfect soundness (Corollary 3) then easily follow from Theorems 1 and 2, as will be found in Subsection 5.2.

### 5.1 Cases with Perfect Completeness

#### 5.1.1 One-Clean-Qubit Case

This subsection proves Theorem 2, stating that any problem computable using logarithmically many clean qubits with one-sided bounded error of perfect completeness and soundness bounded away from one by an inverse-polynomial is necessarily computable using only one clean qubit with perfect completeness and polynomially small soundness error.

*Proof of Theorem 2.* For any polynomial-time computable function  $s: \mathbb{Z}^+ \rightarrow [0, 1]$  satisfying  $1 - s \geq \frac{1}{q}$  for some polynomially bounded function  $q: \mathbb{Z}^+ \rightarrow \mathbb{N}$ , let  $A = (A_{\text{yes}}, A_{\text{no}})$  be a problem in  $\text{Q}_{\log \text{P}}(1, s)$ . Then  $A$  has a polynomial-time uniformly generated family  $\{Q_x\}_{x \in \Sigma^*}$  of quantum circuits such that, for every input  $x$ ,  $p_{\text{acc}}(Q_x, k(|x|)) = 1$  if  $x$  is in  $A_{\text{yes}}$  and  $p_{\text{acc}}(Q_x, k(|x|)) \leq s(|x|)$  if  $x$  is in  $A_{\text{no}}$  for some logarithmically bounded function  $k: \mathbb{Z}^+ \rightarrow \mathbb{N}$ . For any polynomially bounded function  $p: \mathbb{Z}^+ \rightarrow \mathbb{N}$ , the proof constructs a polynomial-time uniformly generated family of quantum circuits that puts  $A$  in  $\text{Q}_{[1]} \text{P}(1, \frac{1}{p})$ .

Fix an input  $x$ .

From the circuit  $Q_x$ , one first constructs a quantum circuit  $R_x$  according to the ONE-CLEAN-QUBIT SIMULATION PROCEDURE. By Proposition 17, it holds that  $p_{\text{acc}}(R_x, 1) = 1$  if  $x$  is in  $A_{\text{yes}}$ , and

$$1 - 2^{-k(|x|)} \leq p_{\text{acc}}(R_x, 1) \leq 1 - 2^{-k(|x|)}(1 - s(|x|)) \leq 1 - \frac{1}{r_1(|x|)}$$

if  $x$  is in  $A_{\text{no}}$ , where  $r_1: \mathbb{Z}^+ \rightarrow \mathbb{N}$  is the polynomially bounded function defined by  $r_1 = 2^k q$ .

From the circuit  $R_x$ , one constructs a quantum circuit  $R'_x$  according to the RANDOMNESS AMPLIFICATION PROCEDURE with the integer  $N = r_1(|x|) r_2(|x|)$  for a polynomially bounded function  $r_2: \mathbb{Z}^+ \rightarrow \mathbb{N}$  satisfying  $r_2 \geq p + 1$ . By Proposition 18, it holds that  $p_{\text{acc}}(R'_x, 1) = 1$  if  $x$  is in  $A_{\text{yes}}$ , and

$$\frac{1}{2} \leq p_{\text{acc}}(R'_x, 1) \leq \frac{1}{2} + \frac{1}{2} \left( 1 - \frac{2}{r_1(|x|)} \right)^{r_1(|x|) r_2(|x|)} < \frac{1}{2} + \frac{1}{2} e^{-2 r_2(|x|)} < \frac{1}{2} + 2^{-2 r_2(|x|)}$$

if  $x$  is in  $A_{\text{no}}$ .

Finally, from the circuit  $R'_x$ , one constructs a quantum circuit  $R''_x$  according to the ONE-CLEAN-QUBIT STABILITY CHECKING PROCEDURE with  $N$  to be the smallest integer that is a power of two and at least  $(6p(|x|))^3$  (i.e.,  $N = 2^{\lceil 3 \log(6p(|x|)) \rceil} > 2^6 = 64$ ). By Propositions 19 and 20, it holds that  $p_{\text{acc}}(R''_x, 1) = 1$  if  $x$  is in  $A_{\text{yes}}$ , and

$$p_{\text{acc}}(R''_x, 1) \leq \frac{1}{2p(|x|)} + 4 \cdot 2^{-2 r_2(|x|)} < \frac{1}{p(|x|)}$$

if  $x$  is in  $A_{\text{no}}$ , where the last inequality follows from the facts that  $r_2 \geq p + 1$  and that the inequality  $2^n > n$  holds.

The claim follows with the polynomial-time uniformly generated family  $\{R''_x\}_{x \in \Sigma^*}$  of quantum circuits.  $\square$

#### 5.1.2 Two-Clean-Qubit Case

This subsection proves Theorem 1, stating that, using two clean qubits rather than one, soundness can be made exponentially small.

*Proof of Theorem 1.* The proof is essentially the same as that of Theorem 2, except that the circuit  $R_x''$  is constructed according to the TWO-CLEAN-QUBIT STABILITY CHECKING PROCEDURE rather than the ONE-CLEAN-QUBIT STABILITY CHECKING PROCEDURE.

For any polynomial-time computable function  $s: \mathbb{Z}^+ \rightarrow [0, 1]$  satisfying  $1 - s \geq \frac{1}{q}$  for some polynomially bounded function  $q: \mathbb{Z}^+ \rightarrow \mathbb{N}$ , let  $A = (A_{\text{yes}}, A_{\text{no}})$  be a problem in  $\text{Q}_{\log}\text{P}(1, s)$ . Then  $A$  has a polynomial-time uniformly generated family  $\{Q_x\}_{x \in \Sigma^*}$  of quantum circuits such that, for every input  $x$ ,  $p_{\text{acc}}(Q_x, k(|x|)) = 1$  if  $x$  is in  $A_{\text{yes}}$  and  $p_{\text{acc}}(Q_x, k(|x|)) \leq s(|x|)$  if  $x$  is in  $A_{\text{no}}$  for some logarithmically bounded function  $k: \mathbb{Z}^+ \rightarrow \mathbb{N}$ . For any polynomially bounded function  $p: \mathbb{Z}^+ \rightarrow \mathbb{N}$ , the proof constructs a polynomial-time uniformly generated family of quantum circuits that puts  $A$  in  $\text{Q}_{[2]}\text{P}(1, 2^{-p})$ .

Fix an input  $x$ .

From the circuit  $Q_x$ , one first constructs a quantum circuit  $R_x$  according to the ONE-CLEAN-QUBIT SIMULATION PROCEDURE. By Proposition 17, it holds that  $p_{\text{acc}}(R_x, 1) = 1$  if  $x$  is in  $A_{\text{yes}}$ , and

$$1 - 2^{-k(|x|)} \leq p_{\text{acc}}(R_x, 1) \leq 1 - 2^{-k(|x|)}(1 - s(|x|)) \leq 1 - \frac{1}{r(|x|)}$$

if  $x$  is in  $A_{\text{no}}$ , where  $r: \mathbb{Z}^+ \rightarrow \mathbb{N}$  is the polynomially bounded function defined by  $r = 2^k q$ .

From the circuit  $R_x$ , one constructs a quantum circuit  $R'_x$  according to the RANDOMNESS AMPLIFICATION PROCEDURE with the integer  $N \geq \alpha r(|x|)$ , where  $\alpha = \frac{3}{2} \ln 2 < 1.04$ . By Proposition 18, it holds that  $p_{\text{acc}}(R'_x, 1) = 1$  if  $x$  is in  $A_{\text{yes}}$ , and

$$\frac{1}{2} \leq p_{\text{acc}}(R'_x, 1) \leq \frac{1}{2} + \frac{1}{2} \left(1 - \frac{2}{r(|x|)}\right)^{\alpha r(|x|)} < \frac{1}{2} + \frac{1}{2} e^{-3 \ln 2} = \frac{1}{2} + \frac{1}{16}$$

if  $x$  is in  $A_{\text{no}}$ .

Finally, from the circuit  $R'_x$ , one constructs a quantum circuit  $R_x''$  according to the TWO-CLEAN-QUBIT STABILITY CHECKING PROCEDURE with  $N$  to be the smallest integer that is a power of two and at least  $16(p(|x|) + 1)$  (i.e.,  $N = 2^{\lceil \log(p(|x|) + 1) \rceil + 4}$ ). If  $x$  is in  $A_{\text{yes}}$ , Proposition 21 ensures that  $p_{\text{acc}}(R_x'', 2) = 1$ . On the other hand, if  $x$  is in  $A_{\text{no}}$ , Proposition 22 ensures that

$$p_{\text{acc}}(R_x'', 2) \leq 2^{-p(|x|)}$$

(notice that the bounds for the probability  $p_{\text{acc}}(R'_x, 1)$  above are such that Proposition 22 can be used to show the upper bound for the probability  $p_{\text{acc}}(R_x'', 2)$ ).

The claim follows with the polynomial-time uniformly generated family  $\{R_x''\}_{x \in \Sigma^*}$  of quantum circuits.  $\square$

## 5.2 Cases with Perfect Soundness

Now Corollary 3 is immediate from Theorems 1 and 2 by considering complement problems.

*Proof of Corollary 3.* For any polynomial-time computable function  $c: \mathbb{Z}^+ \rightarrow [0, 1]$  satisfying  $c \geq \frac{1}{q}$  for some polynomially bounded function  $q: \mathbb{Z}^+ \rightarrow \mathbb{N}$ , let  $A = (A_{\text{yes}}, A_{\text{no}})$  be a problem in  $\text{Q}_{\log}\text{P}(c, 0)$ . Consider the complement problem  $\bar{A} = (A_{\text{no}}, A_{\text{yes}})$  of  $A$ . As  $\bar{A}$  is in  $\text{Q}_{\log}\text{P}(1, \delta)$  for the constant  $\delta = 1 - c$  satisfying  $1 - \delta = c \geq \frac{1}{q}$ , it follows from Theorems 1 and 2 that  $\bar{A}$  is in  $\text{Q}_{[2]}\text{P}(1, 2^{-p})$  and also in  $\text{Q}_{[1]}\text{P}(1, \frac{1}{p})$ , for any polynomially bounded function  $p: \mathbb{Z}^+ \rightarrow \mathbb{N}$ . This implies that  $A$  is in  $\text{Q}_{[2]}\text{P}(1 - 2^{-p}, 0)$  and also in  $\text{Q}_{[1]}\text{P}(1 - \frac{1}{p}, 0)$ , for any polynomially bounded function  $p: \mathbb{Z}^+ \rightarrow \mathbb{N}$ , as desired.  $\square$

## 6 Error Reduction for Two-Sided-Error Cases

This section considers the cases with two-sided bounded error. First, Subsection 6.1 provides one more technical tool, called the OR-TYPE REPETITION PROCEDURE, which aims to simulate the standard error-reduction method of a repetition with an OR-type decision. By combining this procedure with the three procedures presented in Section 4, Theorems 4 and 5 are proved in Subsection 6.2.

### 6.1 OR-Type Repetition Procedure

Again consider any quantum circuit  $Q$  acting over  $w$  qubits, which is supposed to be applied to the  $k$ -clean-qubit initial state  $\rho_{\text{init}}^{(w,k)} = (|0\rangle\langle 0|)^{\otimes k} \otimes (\frac{I}{2})^{\otimes (w-k)}$ . This section presents a procedure, called the OR-TYPE REPETITION PROCEDURE, that constructs another quantum circuit  $R^{(N)}$  from  $Q$  when a positive integer  $N$  is specified. Intuitively, the circuit  $R^{(N)}$  is designed so that, when some additional clean qubits are available other than the  $k$  clean qubits that are used for the  $k$ -clean-qubit computation induced by  $Q$ , one can perform  $N$  attempts of the  $k$ -clean-qubit computation induced by  $Q$  and accept if and only if at least one of these attempts results in acceptance in the original computation of  $Q$ .

If  $Nk$  clean qubits were allowed to use (in addition to one clean qubit as the output qubit), the above procedure is easily constructed by preparing  $N$  copies of the initial state  $\rho_{\text{init}}^{(w,k)}$ , applying  $Q$  to each of these  $N$  initial states, and accepting if and only if at least one of the  $N$  attempts results in acceptance. The only problem in this construction is that the resulting computation requires unallowably many clean qubits in its initial state, as  $N$  is desirably much larger than  $k$  for the purpose of error reduction of the  $k$ -clean-qubit computations.

To realize this OR-type repetition idea, consider the procedure described in Figure 11. Instead of preparing  $N$  copies of the initial state, now the  $k$  qubits in the register  $Q$  that were initially clean are reused for each attempt. In order to reuse the  $k$  clean qubits, the procedure tries to initialize these qubits by applying the inverse of  $Q$  in Step 2.3 (and measuring each qubit in  $Q$  in the computational basis in Step 2.4). Of course, the initialization may not necessarily succeed every time, and the failure of the initialization is also counted as an “acceptance” in that attempt. The underlying idea of this procedure is essentially the same as that in the method in Ref. [Wat01] used for reducing the computation error of one-sided bounded error logarithmic-space quantum Turing machines, where the phase-shift transformation is applied when the initialization attempt fails, instead of counting it as an “acceptance”. It is unclear whether the phase-shift trick in Ref. [Wat01] works in the present case, as one cannot perfectly judge in the present case whether the initialization succeeds or not, due to the existence of polynomially many qubits that are initially in the totally mixed states.

The actual construction of the above simplified procedure uses a single-qubit register  $O$  and an  $l$ -qubit register  $C$  for  $l = \lceil \log N \rceil + 1$ , and further introduces a single-qubit quantum register  $X_j$  and a  $k$ -qubit register  $Y_j$  for each  $j$  in  $\{1, \dots, N\}$ . The qubits in  $O$ ,  $C$ , and  $Q$  are supposed to be initially in state  $|0\rangle$ , and all the other qubits used in the actual procedure are supposed to be initially in the totally mixed state  $I/2$ . The qubit in  $O$  serves as the output qubit, and the content of  $C$  serves as a counter  $C$  of the simplified procedure. Each conditional increment of the counter value is realized either by using the controlled- $U_{+1}(\mathbb{Z}_{2^l})$  transformation  $\Lambda(U_{+1}(\mathbb{Z}_{2^l}))$  or by combining the  $U_{+1}(\mathbb{Z}_{2^l})$  transformation and the  $k$ -controlled- $U_{+1}(\mathbb{Z}_{2^l})$  transformation  $\Lambda^k(U_{+1}(\mathbb{Z}_{2^l}))$ . The decision of acceptance and rejection at Step 3 of the simplified procedure can be simulated by combining NOT transformations and a generalized Toffoli transformation. Each register  $X_j$  is used to simulate the measurement at Step 2.2 while each register  $Y_j$  is used to simulate the measurement at Step 2.4, for each repetition round  $j$  of Step 2. More precisely, for each repetition round  $j$  of Step 2, the measurement of the qubit in  $Q^{(1)}$  in the computational basis at Step 2.2 is replaced by the application of a CNOT transformation to  $(Q^{(1)}, X_j)$ , while the measurements of the qubits in  $Q$  in the computational basis at Step 2.4 is replaced by the applications of CNOT transformations to  $(Q, Y_j)$ . Figure 12 summarizes the actual construction of the OR-TYPE REPETITION PROCEDURE.

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### OR-TYPE REPETITION PROCEDURE — Simplified Description

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1. Prepare a  $k$ -qubit register  $Q$  where all the qubits in  $Q$  are supposed to be initially in state  $|0\rangle$ . For each  $j$  in  $\{1, \dots, k\}$ , let  $Q^{(j)}$  denote the single-qubit quantum register corresponding to the  $j$ th qubit of  $Q$ . Prepare  $(w - k)$ -qubit registers  $R_j$ , for  $j$  in  $\{1, \dots, N\}$ , where all the qubits in  $R_j$  are supposed to be initially in the totally mixed state  $I/2$ . Initialize a counter  $C$  to 0.
  2. For  $j = 1$  to  $N$ , perform the following:
    - 2.1. Apply  $Q$  to  $(Q, R_j)$ .
    - 2.2. Measure the qubit in  $Q^{(1)}$  in the computational basis. If this results in  $|0\rangle$ , increase the counter  $C$  by one.
    - 2.3. Apply  $Q^\dagger$  to  $(Q, R_j)$ .
    - 2.4. Measure each qubit in  $Q$  in the computational basis. If any of these results in  $|1\rangle$ , increase the counter  $C$  by one.
  3. Reject if  $C = 0$ , and accept otherwise.
- 

Figure 11: The OR-TYPE REPETITION PROCEDURE (a simplified description).

**Proposition 23.** *For any quantum circuit  $Q$  and any positive integer  $N$ , let  $R^{(N)}$  be the quantum circuit corresponding to the OR-TYPE REPETITION PROCEDURE induced by  $Q$  and  $N$ . For the acceptance probability  $p_{\text{acc}}(Q, k)$  of the  $k$ -clean-qubit computation induced by  $Q$  and the acceptance probability  $p_{\text{acc}}(R^{(N)}, k + \lceil \log N \rceil + 2)$  of the  $(k + \lceil \log N \rceil + 2)$ -clean-qubit computation induced by  $R^{(N)}$ , it holds that*

$$1 - (1 - p_{\text{acc}}(Q, k))^N \leq p_{\text{acc}}(R^{(N)}, k + \lceil \log N \rceil + 2) \leq 1 - (1 - p_{\text{acc}}(Q, k))^{2N}.$$

*Proof.* For ease of explanations, the proof analyzes the simplified version of the OR-TYPE REPETITION PROCEDURE in Figure 11, which is sufficient for the claim, as the actual construction of the OR-TYPE REPETITION PROCEDURE exactly simulates the simplified version.

First notice that the counter value remains zero when entering Step 3 if and only if the simulation of  $Q$  results in rejection (i.e., the measurement in Step 2.2 results in  $|1\rangle$ ) and the initialization of the qubits in  $Q$  succeeds (i.e., none of the measurements in Step 2.4 results in  $|1\rangle$ ) for all the  $N$  attempts during Step 2. Suppose that the qubits in  $(Q, R_j)$  form the state  $|\psi_r\rangle = |0\rangle^{\otimes k} \otimes |r\rangle$  when entering Step 2.1, for some  $r$  in  $\Sigma^{(w-k)}$ , and let  $p_r$  be the probability defined by

$$p_r = \|\Pi_1 Q |\psi_r\rangle\|^2,$$

where  $\Pi_1$  is the projection operator acting over  $w$  qubits defined by  $\Pi_1 = |1\rangle\langle 1| \otimes I^{\otimes (w-1)}$ . Then, it is clear that the measurement at Step 2.2 results in  $|1\rangle$  with probability exactly  $p_r$ . Hence, by letting  $\Delta_0$  be the projection operator acting over  $w$  qubits defined by  $\Delta_0 = (|0\rangle\langle 0|)^{\otimes k} \otimes I^{\otimes (w-k)}$ , the joint probability that the measurement at Step 2.2 results in  $|1\rangle$  and none of the measurements in Step 2.4 results in  $|1\rangle$  is given by

$$q_r = \|\Delta_0 Q^\dagger \Pi_1 Q |\psi_r\rangle\|^2.$$

To prove the first inequality of the claim, notice that  $q_r = \|\Delta_0 Q^\dagger \Pi_1 Q |\psi_r\rangle\|^2$  is at most  $p_r = \|\Pi_1 Q |\psi_r\rangle\|^2$ . As the acceptance probability  $p_{\text{acc}}(Q, k)$  of the  $k$ -clean-qubit computation induced by  $Q$  is nothing but the expected



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### OR-TYPE REPETITION PROCEDURE

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1. Given a positive integer  $N$ , let  $l = \lceil \log N \rceil + 1$ . Prepare a single-qubit register  $O$ , an  $l$ -qubit register  $C$ , and a  $k$ -qubit register  $Q$ , where all the qubits in  $O$ ,  $C$ , and  $Q$  are supposed to be initially in state  $|0\rangle$ . For each  $i$  in  $\{1, \dots, k\}$ , let  $Q^{(i)}$  denote the single-qubit quantum register corresponding to the  $i$ th qubit of  $Q$ . Prepare a  $(w - k)$ -qubit register  $R_j$ , a single-qubit register  $X_j$ , and a  $k$ -qubit register  $Y_j$ , for each  $j$  in  $\{1, \dots, N\}$ , where all the qubits in  $R_j$ ,  $X_j$ , and  $Y_j$  are supposed to be initially in the totally mixed state  $I/2$ . For each  $i$  in  $\{1, \dots, k\}$  and  $j$  in  $\{1, \dots, N\}$ , let  $Y_j^{(i)}$  denote the single-qubit quantum register corresponding to the  $i$ th qubit of  $Y_j$ .
  2. For  $j = 1$  to  $N$ , perform the following:
    - 2.1. Apply  $Q_x$  to  $(Q, R_j)$ .
    - 2.2. Apply the CNOT transformation to  $(Q^{(1)}, X_j)$  with the qubit in  $Q^{(1)}$  being the control.  
Apply  $U_{+1}(\mathbb{Z}_{2^l})$  to  $C$  if the content of  $Q^{(1)}$  is 0 (this can be realized by first applying  $X$  to  $Q^{(1)}$ , then applying the controlled- $U_{+1}(\mathbb{Z}_{2^l})$  transformation  $\Lambda(U_{+1}(\mathbb{Z}_{2^l}))$  to  $(Q^{(1)}, C)$  with the qubit in  $Q^{(1)}$  being the control, and further applying  $X$  to  $Q^{(1)}$ ).
    - 2.3. Apply  $Q_x^\dagger$  to  $(Q, R_j)$ .
    - 2.4. For each  $i$  in  $\{1, \dots, k\}$ , apply the CNOT transformation to  $(Q^{(i)}, Y_j^{(i)})$  with the qubit in  $Q^{(i)}$  being the control.  
Apply  $U_{+1}(\mathbb{Z}_{2^l})$  to  $C$  if any of the qubits in  $Q$  contains 1 (this can be realized by first applying  $U_{+1}(\mathbb{Z}_{2^l})$  to  $C$ , next applying  $X$  to each of the qubits in  $Q$ , then applying the  $k$ -controlled- $U_{+1}(\mathbb{Z}_{2^l})^\dagger$  transformation  $\Lambda^k(U_{+1}(\mathbb{Z}_{2^l})^\dagger)$  to  $(Q, C)$  with the qubits in  $Q$  being the control, and further applying  $X$  to each of the qubits in  $Q$ ).
  3. Apply  $X$  to  $O$  if any of the qubits in  $C$  contains 1 (this can be realized by first applying  $X$  to each of the qubits in  $O$  and  $C$ , and then applying the generalized Toffoli transformation  $\Lambda^l(X)$  to  $(C, O)$  with the qubit in  $O$  being the target — and further applying  $X$  to each of the qubits in  $C$  to precisely realize the transformation in this step, which is unnecessary for the purpose of this procedure).  
Measure the qubit in  $O$  in the computational basis. Accept if this results in  $|0\rangle$ , and reject otherwise.
- 

Figure 12: The OR-TYPE REPETITION PROCEDURE.

value of  $1 - p_r$  over  $r$  in  $\Sigma^{(w-k)}$ , it holds that

$$2^{-(w-k)} \sum_{r \in \Sigma^{(w-k)}} q_r \leq 2^{-(w-k)} \sum_{r \in \Sigma^{(w-k)}} p_r = 1 - p_{\text{acc}}(Q, k),$$

and thus, the probability is at most  $1 - p_{\text{acc}}(Q, k)$  for the event that the counter value remains zero after one iteration of Step 2, conditioned that the counter value was zero when starting that iteration. Overall, the probability that the counter value remains zero when entering Step 3 is at most  $(1 - p_{\text{acc}}(Q, k))^N$ , and thus, the procedure results in acceptance with probability at least  $1 - (1 - p_{\text{acc}}(Q, k))^N$ , which shows the first inequality.

For the second inequality of the claim, note that  $\Delta_0 = (|0\rangle\langle 0|)^{\otimes k} \otimes \sum_{r \in \Sigma^{(w-k)}} |r\rangle\langle r| = \sum_{r \in \Sigma^{(w-k)}} |\psi_r\rangle\langle \psi_r|$ , and thus, it holds that

$$q_r \geq \left\| |\psi_r\rangle\langle \psi_r| Q^\dagger \Pi_1 Q |\psi_r\rangle \right\|^2 = \left| \langle \psi_r | Q^\dagger \Pi_1 Q | \psi_r \rangle \right|^2 = \left\| \Pi_1 Q |\psi_r\rangle \right\|^4 = p_r^2.$$

Again using the fact that  $p_{\text{acc}}(Q, k)$  is nothing but the expected value of  $1 - p_r$  over  $r$  in  $\Sigma^{(w-k)}$ , it follows that

$$\begin{aligned} 2^{-(w-k)} \sum_{r \in \Sigma^{(w-k)}} q_r &\geq 2^{-(w-k)} \sum_{r \in \Sigma^{(w-k)}} p_r^2 \\ &\geq \left( 2^{-(w-k)} \sum_{r \in \Sigma^{(w-k)}} p_r \right)^2 = \left( 1 - 2^{-(w-k)} \sum_{r \in \Sigma^{(w-k)}} (1 - p_r) \right)^2 = (1 - p_{\text{acc}}(Q, k))^2, \end{aligned}$$

and thus, the probability is at least  $(1 - p_{\text{acc}}(Q, k))^2$  for the event that the counter value remains zero after one iteration of Step 2, conditioned that the counter value was zero when starting that iteration. Overall, the probability that the counter value remains zero when entering Step 3 is at least  $[(1 - p_{\text{acc}}(Q, k))^2]^N = (1 - p_{\text{acc}}(Q, k))^{2N}$ . Accordingly, the procedure results in acceptance with probability at most  $1 - (1 - p_{\text{acc}}(Q, k))^{2N}$ , and the second inequality follows.  $\square$

## 6.2 Proofs of Theorems 4 and 5

Now we are ready to prove Theorems 4 and 5. First, we prove two lemmas.

**Lemma 24.** *For any constants  $c$  and  $s$  in  $\mathbb{R}$  satisfying  $0 < s < c < 1$ ,*

$$Q_{\log}P(c, s) \subseteq Q_{[1]}P\left(\frac{3}{4}, \frac{5}{8}\right).$$

*Proof.* For any constants  $c$  and  $s$  in  $\mathbb{R}$  satisfying  $0 < s < c < 1$ , let  $A = (A_{\text{yes}}, A_{\text{no}})$  be a problem in  $Q_{\log}P(c, s)$ . Then  $A$  has a polynomial-time uniformly generated family  $\{Q_x\}_{x \in \Sigma^*}$  of quantum circuits such that, for every input  $x$ ,  $p_{\text{acc}}(Q_x, k_1(|x|)) \geq c$  if  $x$  is in  $A_{\text{yes}}$  and  $p_{\text{acc}}(Q_x, k_1(|x|)) \leq s$  if  $x$  is in  $A_{\text{no}}$  for some logarithmically bounded function  $k_1: \mathbb{Z}^+ \rightarrow \mathbb{N}$ . The proof constructs a polynomial-time uniformly generated family of quantum circuits that puts  $A$  in  $Q_{[1]}P(\frac{3}{4}, \frac{5}{8})$ .

Fix an input  $x$ .

From the circuit  $Q_x$ , one first constructs a quantum circuit  $R_x$  that runs  $N$  attempts of  $Q_x$  in parallel, and accepts if and only if at least  $\frac{c+s}{2}$  fraction of the  $N$  attempts results in acceptance. Notice that this can be easily implementable if  $N$  is a power of two, by combining the increment transformation with the threshold-check transformation discussed in Subsection 3.3. By taking a sufficiently large constant  $N$  that is a power of two, it holds that  $p_{\text{acc}}(R_x, k_2(|x|)) \geq \frac{15}{16}$  if  $x$  is in  $A_{\text{yes}}$ , and  $p_{\text{acc}}(R_x, k_2(|x|)) \leq \frac{1}{16}$  if  $x$  is in  $A_{\text{no}}$ , where  $k_2: \mathbb{Z}^+ \rightarrow \mathbb{N}$  is the logarithmically bounded function defined by  $k_2 = Nk_1 + \log N + 1$ .

From the circuit  $R_x$ , one constructs a quantum circuit  $R'_x$  according to the ONE-CLEAN-QUBIT SIMULATION PROCEDURE with  $k = k_2$ . By Proposition 17, it holds that  $p_{\text{acc}}(R'_x, 1) \geq 1 - \frac{31}{256q(|x|)} > 1 - \frac{1}{8q(|x|)}$  if  $x$  is in  $A_{\text{yes}}$ , and  $p_{\text{acc}}(R'_x, 1) \leq 1 - \frac{15}{16q(|x|)}$  if  $x$  is in  $A_{\text{no}}$ , where  $q: \mathbb{Z}^+ \rightarrow \mathbb{N}$  is the polynomially bounded function defined by  $q = 2^{k_2}$ .

Finally, from the circuit  $R'_x$ , one constructs a quantum circuit  $R''_x$  according to the RANDOMNESS AMPLIFICATION PROCEDURE with the integer  $N = 2q(|x|)$ . By Proposition 18, it holds that

$$p_{\text{acc}}(R''_x, 1) \geq \frac{1}{2} + \frac{1}{2} \left( 1 - \frac{1}{4q(|x|)} \right)^{2q(|x|)} > \frac{1}{2} + \frac{1}{2} \left( 1 - 2q(|x|) \cdot \frac{1}{4q(|x|)} \right) = \frac{3}{4}$$

if  $x$  is in  $A_{\text{yes}}$ , and

$$p_{\text{acc}}(R''_x, 1) \leq \frac{1}{2} + \frac{1}{2} \left( 1 - \frac{15}{8q(|x|)} \right)^{2q(|x|)} < \frac{1}{2} + \frac{1}{2} \left( 1 - \frac{15}{8q(|x|)} \right)^{\frac{16}{15}q(|x|)} < \frac{1}{2} + \frac{1}{2e^2} < \frac{5}{8}$$

if  $x$  is in  $A_{\text{no}}$ .

The claim follows with the polynomial-time uniformly generated family  $\{R''_x\}_{x \in \Sigma^*}$  of quantum circuits.  $\square$

**Lemma 25.** For any constants  $c$  and  $s$  in  $\mathbb{R}$  satisfying  $0 < s < c < 1$  and for any polynomially bounded function  $p: \mathbb{Z}^+ \rightarrow \mathbb{N}$ ,

$$Q_{[1]}P(c, s) \subseteq Q_{\log}P\left(1 - 2^{-p}, \frac{1}{2}\right).$$

*Proof.* For any constants  $c$  and  $s$  in  $\mathbb{R}$  satisfying  $0 < s < c < 1$ , let  $A = (A_{\text{yes}}, A_{\text{no}})$  be a problem in  $Q_{[1]}P(c, s)$ . Then  $A$  has a polynomial-time uniformly generated family  $\{Q_x\}_{x \in \Sigma^*}$  of quantum circuits such that, for every input  $x$ ,  $p_{\text{acc}}(Q_x, 1) \geq c$  if  $x$  is in  $A_{\text{yes}}$  and  $p_{\text{acc}}(Q_x, 1) \leq s$  if  $x$  is in  $A_{\text{no}}$ . For any polynomially bounded function  $p: \mathbb{Z}^+ \rightarrow \mathbb{N}$ , the proof constructs a polynomial-time uniformly generated family of quantum circuits that puts  $A$  in  $Q_{\log}P(1 - 2^{-p}, \frac{1}{2})$ .

Fix an input  $x$ .

From the circuit  $Q_x$ , one first constructs a quantum circuit  $R_x$  that runs  $l(|x|)$  attempts of  $Q_x$  in parallel, and accepts if and only if at least  $\frac{c+s}{2}$  fraction of the  $l(|x|)$  attempts results in acceptance, where  $l: \mathbb{Z}^+ \rightarrow \mathbb{N}$  is some logarithmically bounded function such that  $l(n)$  is a power of two for every  $n$  in  $\mathbb{Z}^+$ . Again notice that this can be easily implementable by combining the increment transformation with the threshold-check transformation discussed in Subsection 3.3. By taking such a function  $l$  that is at least  $\frac{2}{(c-s)^2}(\log p + 2)$ , it holds that  $p_{\text{acc}}(R_x, k(|x|)) \geq 1 - \frac{1}{4p(|x|)}$  if  $x$  is in  $A_{\text{yes}}$ , and  $p_{\text{acc}}(R_x, k(|x|)) \leq \frac{1}{4p(|x|)}$  if  $x$  is in  $A_{\text{no}}$ , where  $k: \mathbb{Z}^+ \rightarrow \mathbb{N}$  is the logarithmically bounded function defined by  $k = l + \log l + 1$ .

From the circuit  $R_x$ , one further constructs a quantum circuit  $R'_x$  according to the OR-TYPE REPETITION PROCEDURE with the integer  $N = p(|x|)$ . By Proposition 23, it holds that

$$p_{\text{acc}}(R'_x, k(|x|) + \lceil \log p(|x|) \rceil + 2) \geq 1 - \left(\frac{1}{4p(|x|)}\right)^{p(|x|)} > 1 - 2^{-p(|x|)}$$

if  $x$  is in  $A_{\text{yes}}$ , and

$$p_{\text{acc}}(R'_x, k(|x|) + \lceil \log p(|x|) \rceil + 2) < 1 - \left(1 - \frac{1}{4p(|x|)}\right)^{2p(|x|)} < 1 - \left(1 - 2p(|x|) \cdot \frac{1}{4p(|x|)}\right) = \frac{1}{2}$$

if  $x$  is in  $A_{\text{no}}$ .

The claim follows with the polynomial-time uniformly generated family  $\{R'_x\}_{x \in \Sigma^*}$  of quantum circuits.  $\square$

### 6.2.1 One-Clean-Qubit Case

By using Lemmas 24 and 25, Theorem 5 is proved as follows.

*Proof of Theorem 5.* For any constants  $c$  and  $s$  in  $\mathbb{R}$  satisfying  $0 < s < c < 1$ , let  $A = (A_{\text{yes}}, A_{\text{no}})$  be a problem in  $Q_{\log}P(c, s)$ . It is sufficient to show that the containment  $Q_{\log}P(c, s) \subseteq Q_{[1]}P(1 - 2^{-p}, \frac{1}{p})$  holds for any polynomially bounded function  $p: \mathbb{Z}^+ \rightarrow \mathbb{N}$ . Indeed, the containment  $Q_{\log}P(c, s) \subseteq Q_{[1]}P(1 - \frac{1}{p}, 2^{-p})$  is then proved by considering the complement problem  $\bar{A} = (A_{\text{no}}, A_{\text{yes}})$  of  $A$ : As  $\bar{A}$  is in  $Q_{\log}P(\varepsilon, \delta)$  for the constants  $\varepsilon = 1 - s$  and  $\delta = 1 - c$  satisfying  $0 < \delta < \varepsilon < 1$ , the containment above to be proved implies that  $\bar{A}$  is in  $Q_{[1]}P(1 - 2^{-p}, \frac{1}{p})$ , and thus, that  $A$  is in  $Q_{[1]}P(1 - \frac{1}{p}, 2^{-p})$ , for any polynomially bounded function  $p: \mathbb{Z}^+ \rightarrow \mathbb{N}$ .

From Lemma 24,  $A$  is in  $Q_{[1]}P(\frac{3}{4}, \frac{5}{8})$ . Therefore, from Lemma 25,  $A$  is in  $Q_{\log}P(1 - 2^{-q}, \frac{1}{2})$ , where  $q: \mathbb{Z}^+ \rightarrow \mathbb{N}$  is the polynomially bounded function defined by  $q = p + 4\lceil \log p \rceil + 13$ . Hence,  $A$  has a polynomial-time uniformly generated family  $\{Q_x\}_{x \in \Sigma^*}$  of quantum circuits such that, for every input  $x$ ,  $p_{\text{acc}}(Q_x, k(|x|)) \geq 1 - 2^{-q(|x|)}$  if  $x$  is in  $A_{\text{yes}}$  and  $p_{\text{acc}}(Q_x, k(|x|)) \leq \frac{1}{2}$  if  $x$  is in  $A_{\text{no}}$ , for some logarithmically bounded function  $k: \mathbb{Z}^+ \rightarrow \mathbb{N}$ . The rest of the proof is essentially the same as the proof of Theorem 2 in Section 5.

Fix an input  $x$ .

From the circuit  $Q_x$ , one first constructs a quantum circuit  $R_x$  according to the ONE-CLEAN-QUBIT SIMULATION PROCEDURE. By Proposition 17, it holds that

$$p_{\text{acc}}(R_x, 1) \geq 1 - 2^{-k(|x|)} \left[ 1 - (1 - 2^{-q(|x|)})^2 \right] > 1 - 2^{-q(|x|) - k(|x|) + 1} = 1 - \frac{2^{-q(|x|) + 2}}{r_1(|x|)}$$

if  $x$  is in  $A_{\text{yes}}$ , and

$$1 - \frac{2}{r(|x|)} = 1 - 2^{-k(|x|)} \leq p_{\text{acc}}(R_x, 1) \leq 1 - 2^{-k(|x|) - 1} = 1 - \frac{1}{r_1(|x|)}$$

if  $x$  is in  $A_{\text{no}}$ , where  $r_1: \mathbb{Z}^+ \rightarrow \mathbb{N}$  is the polynomially bounded function defined by  $r_1 = 2^{k+1}$ .

From the circuit  $R_x$ , one constructs a quantum circuit  $R'_x$  according to the RANDOMNESS AMPLIFICATION PROCEDURE with the integer  $N = r_1(|x|) r_2(|x|)$  for a polynomially bounded function  $r_2: \mathbb{Z}^+ \rightarrow \mathbb{N}$  such that  $r_2 = p + 1$ . By Proposition 18, it holds that

$$\begin{aligned} p_{\text{acc}}(R'_x, 1) &\geq \frac{1}{2} + \frac{1}{2} \left( 1 - \frac{2^{-q(|x|) + 3}}{r_1(|x|)} \right)^{r_1(|x|) r_2(|x|)} \\ &> \frac{1}{2} + \frac{1}{2} \left( 1 - r_1(|x|) r_2(|x|) \cdot \frac{2^{-q(|x|) + 3}}{r_1(|x|)} \right) = 1 - r_2(|x|) \cdot 2^{-q(|x|) + 2} \end{aligned}$$

if  $x$  is in  $A_{\text{yes}}$ , while

$$\frac{1}{2} \leq p_{\text{acc}}(R'_x, 1) \leq \frac{1}{2} + \frac{1}{2} \left( 1 - \frac{2}{r_1(|x|)} \right)^{r_1(|x|) r_2(|x|)} < \frac{1}{2} + \frac{1}{2} e^{-2r_2(|x|)} < \frac{1}{2} + 2^{-2r_2(|x|)}$$

if  $x$  is in  $A_{\text{no}}$ .

Finally, from the circuit  $R'_x$ , one constructs a quantum circuit  $R''_x$  according to the ONE-CLEAN-QUBIT STABILITY CHECKING PROCEDURE with  $N$  to be the smallest integer that is a power of two and at least  $(6p(|x|))^3$  (i.e.,  $N = 2^{\lceil 3 \log(6p(|x|)) \rceil} > 2^6 = 64$ ). If  $x$  is in  $A_{\text{yes}}$ , Proposition 19 ensures that

$$\begin{aligned} p_{\text{acc}}(R''_x, 1) &\geq \left( 1 - r_2(|x|) \cdot 2^{-q(|x|) + 2} \right)^{4 \cdot (6p(|x|))^3 - 1} \\ &> \left( 1 - r_2(|x|) \cdot 2^{-q(|x|) + 2} \right)^{2 \cdot (8p(|x|))^3} > 1 - 2 \cdot (8p(|x|))^3 \cdot r_2(|x|) \cdot 2^{-q(|x|) + 2} \geq 1 - 2^{-p(|x|)}, \end{aligned}$$

where the first inequality follows from the fact that  $N$  is at most  $2 \cdot (6p(|x|))^3$ , the second inequality follows from the fact that  $2 \cdot (6p(|x|))^3 < (8p(|x|))^3$ , and the last inequality follows from the fact that  $q(|x|) = p(|x|) + 4 \lceil \log p(|x|) \rceil + 13$ , that  $(8p(|x|))^3 = 2^{3 \log(8p(|x|))} = 2^{3 \log p(|x|) + 9}$ , that  $r_2(|x|) = p(|x|) + 1 = 2^{\log(p(|x|) + 1)}$ , and that the inequality  $\log(n + 1) \leq \log n + 1$  holds for any  $n \geq 1$ . On the other hand, if  $x$  is in  $A_{\text{no}}$ , Proposition 20, ensures that

$$p_{\text{acc}}(R''_x, 1) \leq \frac{1}{2p(|x|)} + 4 \cdot 2^{-2r_2(|x|)} < \frac{1}{p(|x|)},$$

where the last inequality follows from the facts that  $r_2 \geq p + 1$  and that the inequality  $2^n > n$  holds.

The claim follows with the polynomial-time uniformly generated family  $\{R''_x\}_{x \in \Sigma^*}$  of quantum circuits.  $\square$

### 6.2.2 Two-Clean-Qubit Case

Again by using Lemmas 24 and 25, Theorem 4 is proved as follows.

*Proof of Theorem 4.* The proof is essentially the same as that of Theorem 5, except that the circuit  $R_x''$  is constructed according to the TWO-CLEAN-QUBIT STABILITY CHECKING PROCEDURE rather than the ONE-CLEAN-QUBIT STABILITY CHECKING PROCEDURE.

For any constants  $c$  and  $s$  in  $\mathbb{R}$  satisfying  $0 < s < c < 1$ , let  $A = (A_{\text{yes}}, A_{\text{no}})$  be a problem in  $\text{Q}_{\log}P(c, s)$ . From Lemma 24,  $A$  is in  $\text{Q}_{[1]}P(\frac{3}{4}, \frac{5}{8})$ . Therefore, from Lemma 25,  $A$  is in  $\text{Q}_{\log}P(1 - 2^{-q}, \frac{1}{2})$ , where  $q: \mathbb{Z}^+ \rightarrow \mathbb{N}$  is the polynomially bounded function defined by  $q = p + \lceil \log p \rceil + 12$ . Hence,  $A$  has a polynomial-time uniformly generated family  $\{Q_x\}_{x \in \Sigma^*}$  of quantum circuits such that, for every input  $x$ ,  $p_{\text{acc}}(Q_x, k(|x|)) \geq 1 - 2^{-q(|x|)}$  if  $x$  is in  $A_{\text{yes}}$  and  $p_{\text{acc}}(Q_x, k(|x|)) \leq \frac{1}{2}$  if  $x$  is in  $A_{\text{no}}$ , for some logarithmically bounded function  $k: \mathbb{Z}^+ \rightarrow \mathbb{N}$ . The rest of the proof is essentially the same as the proof of Theorem 1 in Section 5.

Fix an input  $x$ .

From the circuit  $Q_x$ , one first constructs a quantum circuit  $R_x$  according to the ONE-CLEAN-QUBIT SIMULATION PROCEDURE. As in the proof of Theorem 5, by Proposition 17, it holds that

$$p_{\text{acc}}(R_x, 1) > 1 - \frac{2^{-q(|x|)+2}}{r(|x|)}$$

if  $x$  is in  $A_{\text{yes}}$ , and

$$1 - \frac{2}{r(|x|)} \leq p_{\text{acc}}(R_x, 1) \leq 1 - \frac{1}{r(|x|)}$$

if  $x$  is in  $A_{\text{no}}$ , where  $r: \mathbb{Z}^+ \rightarrow \mathbb{N}$  is the polynomially bounded function defined by  $r = 2^{k+1}$ .

From the circuit  $R_x$ , one constructs a quantum circuit  $R'_x$  according to the RANDOMNESS AMPLIFICATION PROCEDURE with the integer  $N \geq \alpha r(|x|)$ , where  $\alpha = \frac{3}{2} \ln 2 < 1.04$ . By Proposition 18, it holds that

$$\begin{aligned} p_{\text{acc}}(R'_x, 1) &\geq \frac{1}{2} + \frac{1}{2} \left( 1 - \frac{2^{-q(|x|)+3}}{r(|x|)} \right)^{\alpha r(|x|)} \\ &> \frac{1}{2} + \frac{1}{2} \left( 1 - \alpha r(|x|) \cdot \frac{2^{-q(|x|)+3}}{r(|x|)} \right) = 1 - \alpha \cdot 2^{-q(|x|)+2} \end{aligned}$$

if  $x$  is in  $A_{\text{yes}}$ , while

$$\frac{1}{2} \leq p_{\text{acc}}(R'_x, 1) \leq \frac{1}{2} + \frac{1}{2} \left( 1 - \frac{2}{r(|x|)} \right)^{\alpha r(|x|)} < \frac{1}{2} + \frac{1}{2} e^{-3 \ln 2} = \frac{1}{2} + \frac{1}{16}$$

if  $x$  is in  $A_{\text{no}}$ .

Finally, from the circuit  $R'_x$ , one constructs a quantum circuit  $R''_x$  according to the TWO-CLEAN-QUBIT STABILITY CHECKING PROCEDURE with  $N$  to be the smallest integer that is a power of two and at least  $16(p(|x|) + 1)$  (i.e.,  $N = 2^{\lceil \log(p(|x|)+1) \rceil + 4}$ ). If  $x$  is in  $A_{\text{yes}}$ , Proposition 21 ensures that

$$p_{\text{acc}}(R''_x, 2) \geq (1 - \alpha \cdot 2^{-q(|x|)+2})^{256(p(|x|)+1)} > 1 - 256(p(|x|) + 1) \cdot \alpha \cdot 2^{-q(|x|)+2} > 1 - 2^{-p(|x|)},$$

where the first inequality follows from the fact that  $N$  is at most  $2 \cdot 16(p(|x|) + 1) = 32(p(|x|) + 1)$ , and the last inequality follows from the fact that  $q(|x|) = p(|x|) + \lceil \log p(|x|) \rceil + 12$ , that  $\alpha = \frac{3}{2} \ln 2 < 1.04 < 2$ , that  $p(|x|) + 1 = 2^{\log(p(|x|)+1)}$ , and that the inequality  $\log(n+1) \leq \log n + 1$  holds for any  $n \geq 1$ .

On the other hand, if  $x$  is in  $A_{\text{no}}$ , Proposition 22 ensures that

$$p_{\text{acc}}(R''_x, 2) \leq 2^{-p(|x|)}$$

(notice that the bounds for the probability  $p_{\text{acc}}(R'_x, 1)$  above are such that Proposition 22 can be used to show the upper bound for the probability  $p_{\text{acc}}(R''_x, 2)$ ).

The claim follows with the polynomial-time uniformly generated family  $\{R''_x\}_{x \in \Sigma^*}$  of quantum circuits.  $\square$

## 7 Completeness Results for TRACE ESTIMATION Problem

This section proves Theorem 6, which states that the TRACE ESTIMATION problem  $\text{TREST}(a, b)$  with any parameters  $a$  and  $b$  satisfying  $0 < b < a < 1$  is complete for both  $\text{BQ}_{\log}\text{P}$  and  $\text{BQ}_{[1]}\text{P}$  under polynomial-time many-one reduction.

As presented in Ref. [SJ08], the following two properties are known to hold.

**Lemma 26.** *For any constants  $a$  and  $b$  in  $\mathbb{R}$  satisfying  $-1 \leq b < a \leq 1$ ,  $\text{TREST}(a, b)$  is in  $\text{Q}_{[1]}\text{P}(\frac{1}{2} + \frac{a}{2}, \frac{1}{2} + \frac{b}{2})$ .*

**Lemma 27.** *For any constants  $a$  and  $b$  in  $\mathbb{R}$  satisfying  $0 \leq b < a \leq 1$ ,  $\text{TREST}(a, b)$  is hard for  $\text{Q}_{[1]}\text{P}(\frac{a}{4}, \frac{b}{4})$  under polynomial-time many-one reduction.*

Theorem 6 is then easily proved as follows, by combining Theorem 5 and these two lemmas.

*Proof of Theorem 6.* We show the  $\text{BQ}_{[1]}\text{P}$ -completeness of  $\text{TREST}(a, b)$ . The  $\text{BQ}_{\log}\text{P}$ -completeness is then trivial, as  $\text{BQ}_{\log}\text{P} = \text{BQ}_{[1]}\text{P}$  due to Theorem 5. In what follows, fix the parameters  $a$  and  $b$ ,  $0 < b < a < 1$ , of  $\text{TREST}(a, b)$ .

For the membership that  $\text{TREST}(a, b)$  is in  $\text{BQ}_{[1]}\text{P}$ , notice that the class  $\text{Q}_{[1]}\text{P}(\frac{1}{2} + \frac{a}{2}, \frac{1}{2} + \frac{b}{2})$  is included in the class  $\text{Q}_{[1]}\text{P}(c, s)$  for any constants  $c$  and  $s$  satisfying that  $c \geq \max\{\frac{2}{3}, \frac{1}{2} + \frac{a}{2}\}$  and  $s \leq \min\{\frac{1}{3}, \frac{1}{2} + \frac{b}{2}\}$ , due to Theorem 5. Therefore, the class  $\text{Q}_{[1]}\text{P}(\frac{1}{2} + \frac{a}{2}, \frac{1}{2} + \frac{b}{2})$  is included in  $\text{BQ}_{[1]}\text{P}$  also. Hence, the membership is immediate, as Lemma 26 ensures that  $\text{TREST}(a, b)$  is in  $\text{Q}_{[1]}\text{P}(\frac{1}{2} + \frac{a}{2}, \frac{1}{2} + \frac{b}{2})$ .

Now for the  $\text{BQ}_{[1]}\text{P}$ -hardness of  $\text{TREST}(a, b)$ , notice that  $\text{BQ}_{[1]}\text{P}$  is included in the class  $\text{Q}_{[1]}\text{P}(c, s)$  for any constants  $c$  and  $s$  satisfying that  $c \geq \max\{\frac{2}{3}, \frac{a}{4}\}$  and  $s \leq \min\{\frac{1}{3}, \frac{b}{4}\}$ , due to Theorem 5. Therefore,  $\text{BQ}_{[1]}\text{P}$  is included in the class  $\text{Q}_{[1]}\text{P}(\frac{a}{4}, \frac{b}{4})$  also. From Lemma 27,  $\text{TREST}(a, b)$  is hard for  $\text{Q}_{[1]}\text{P}(\frac{a}{4}, \frac{b}{4})$  under polynomial-time many-one reduction, and thus,  $\text{TREST}(a, b)$  is hard for  $\text{BQ}_{[1]}\text{P}$  also, under polynomial-time many-one reduction.  $\square$

*Remark.* As mentioned in Subsection 3.5, the completeness results of Theorem 6 hold even under logarithmic-space many-one reduction, if the classes  $\text{BQ}_{\log}\text{P}$  and  $\text{BQ}_{[1]}\text{P}$  are defined with logarithmic-space uniformly generated family of quantum circuits.

## 8 Hardness of Weak Classical Simulations of DQC1 Computation

This section deals with the hardness of weakly simulating a DQC1 computation. First, Subsection 8.1 reviews the notions of weak simulatability that are discussed in this paper. Subsection 8.2 then proves Theorems 7 and 8.

### 8.1 Weak Simulatability

Following conventions, this paper uses the following notions of simulatability.

Consider any family  $\{Q_x\}_{x \in \Sigma^*}$  of quantum circuits. For each circuit  $Q_x$ , suppose that  $m$  output qubits are measured in the computational basis after the application of  $Q_x$  to a certain prescribed initial state (which will be clear from the context). Let  $P_x: \Sigma^m \rightarrow [0, 1]$  be the probability distribution derived from the output of  $Q_x$  (i.e.,  $P_x(y_1, \dots, y_m)$  is the probability of obtaining the measurement result  $(y_1, \dots, y_m)$  in  $\Sigma^m$  when  $Q_x$  is applied to the prescribed initial state).

The family  $\{Q_x\}_{x \in \Sigma^*}$  is *weakly simulatable with multiplicative error  $c \geq 1$*  if there exists a family  $\{P'_x\}_{x \in \Sigma^*}$  of probability distributions that can be sampled classically in polynomial time such that, for any  $x$  in  $\Sigma^*$  and any  $(y_1, \dots, y_m)$  in  $\Sigma^m$ ,

$$\frac{1}{c} P_x(y_1, \dots, y_m) \leq P'_x(y_1, \dots, y_m) \leq c P_x(y_1, \dots, y_m). \quad (2)$$

Similarly, the family  $\{Q_x\}_{x \in \Sigma^*}$  is *weakly simulatable with exponentially small additive error* if, for any polynomially bounded function  $q$ , there exists a family  $\{P'_x\}_{x \in \Sigma^*}$  of probability distributions that can be sampled classically in polynomial time such that, for any  $x$  in  $\Sigma^*$  and any  $(y_1, \dots, y_m)$  in  $\Sigma^m$ ,

$$|P_x(y_1, \dots, y_m) - P'_x(y_1, \dots, y_m)| \leq 2^{-q(|x|)}.$$

A few remarks are in order regarding the notions of weak simulatability above.

First, the notion of weak simulatability with multiplicative error was first defined in Ref. [TD04] in a slightly different form. The definition taken in this paper is found in Refs. [BJS11, MFF14], for instance. The version in Ref. [TD04] uses the bound  $|P_x(y_1, \dots, y_m) - P'_x(y_1, \dots, y_m)| \leq \varepsilon P_x(y_1, \dots, y_m)$  instead of the bounds (2), and these two versions are essentially equivalent. The results in this paper hold for any  $\varepsilon$  in  $[0, 1)$  when using the version in Ref. [TD04].

The notion of weak simulatability with exponentially small additive error was introduced in Ref. [TYT14], and was used also in Ref. [TTYT15].

As the notion of weak simulatability with multiplicative error is often used when discussing the classical simulatability of quantum models [TD04, BJS11, AA13, NVdN13, JVdN14, MFF14, Bro15], the hardness result on the DQC1 model under this notion certainly makes it possible to discuss the power of the DQC1 model along the line of these existing studies. As discussed in Refs. [BJS11, AA13], however, a much more reasonable notion is the weak simulatability with *polynomially small additive error* in total variation distance. Proving or disproving classical simulatability under this notion is one of the most important open problems in most of quantum computation models including the DQC1 model.

## 8.2 Proofs of Theorems 7 and 8

First, Theorem 7, stating that the restriction to the DQC1 computation does not change the complexity classes NQP and SBQP, can be easily proved by using the ONE-CLEAN-QUBIT SIMULATION PROCEDURE presented in Subsection 4.1.

*Proof of Theorem 7.* It suffices to show that  $\text{co-NQP} \subseteq \text{co-NQ}_{[1]}P$  and  $\text{co-SBQP} \subseteq \text{co-SBQ}_{[1]}P$ .

We first show that  $\text{co-NQP} \subseteq \text{co-NQ}_{[1]}P$ .

Consider any problem  $A = (A_{\text{yes}}, A_{\text{no}})$  is in  $\text{co-NQP}$ , and let  $\{Q_x\}_{x \in \Sigma^*}$  be a polynomial-time uniformly generated family of quantum circuits that witnesses this fact. For each  $x$  in  $\Sigma^*$ , the circuit  $Q_x$  acts over  $w(|x|)$  qubits, for some polynomially bounded function  $w: \mathbb{Z}^+ \rightarrow \mathbb{N}$ . By the definition of NQP, for every input  $x$ , the acceptance probability  $p_{\text{acc}}(Q_x, w(|x|))$  of the circuit  $Q_x$  is one if  $x$  is in  $A_{\text{yes}}$ , while it is less than one if  $x$  is in  $A_{\text{no}}$ .

Fix an input  $x$ , and consider the quantum circuit  $R_x$  corresponding to the ONE-CLEAN-QUBIT SIMULATION PROCEDURE induced by  $Q_x$  and  $w(|x|)$ . By the properties of  $Q_x$ , Proposition 17 ensures that the acceptance probability  $p_{\text{acc}}(R_x, 1)$  of the one-clean-qubit computation induced by  $R_x$  is one if  $x$  is in  $A_{\text{yes}}$ , while it is less than one if  $x$  is in  $A_{\text{no}}$ , which implies that  $A$  is in  $\text{co-NQ}_{[1]}P$ .

Now we show that  $\text{co-SBQP} \subseteq \text{co-SBQ}_{[1]}P$ .

Consider any problem  $A = (A_{\text{yes}}, A_{\text{no}})$  is in  $\text{co-SBQP}$ . Then, by the amplification property of SBQP presented in Ref. [Kup15], there exists a polynomial-time uniformly generated family  $\{Q_x\}_{x \in \Sigma^*}$  of quantum circuits such that, for every input  $x$ ,  $Q_x$  acts over  $w(|x|)$  qubits, for some polynomially bounded function  $w: \mathbb{Z}^+ \rightarrow \mathbb{N}$ , and the acceptance probability  $p_{\text{acc}}(Q_x, w(|x|))$  of the circuit  $Q_x$  is at least  $1 - 2^{-p(|x|)-2}$  if  $x$  is in  $A_{\text{yes}}$ , while it is at most  $1 - 2^{-p(|x|)}$  if  $x$  is in  $A_{\text{no}}$ , for some polynomially bounded function  $p: \mathbb{Z}^+ \rightarrow \mathbb{N}$ .

Fix an input  $x$ , and consider the quantum circuit  $R_x$  corresponding to the ONE-CLEAN-QUBIT SIMULATION PROCEDURE induced by  $Q_x$  and  $w(|x|)$ . By the properties of  $Q_x$ , Proposition 17 ensures that the acceptance probability  $p_{\text{acc}}(R_x, 1)$  of the one-clean-qubit computation induced by  $R_x$  is at least

$$1 - 2^{-w(|x|)} \left[ 1 - (1 - 2^{-p(|x|)-2})^2 \right] > 1 - 2^{-w(|x|)} \left[ 1 - (1 - 2^{-p(|x|)-1}) \right] = 1 - 2^{-w(|x|)-p(|x|)-1},$$

if  $x$  is in  $A_{\text{yes}}$ , while it is at most

$$1 - 2^{-w(|x|)} [1 - (1 - 2^{-p(|x|)})] = 1 - 2^{-w(|x|) - p(|x|)},$$

if  $x$  is in  $A_{\text{no}}$ , which ensures that  $A$  is in  $\text{co-SBQ}_{[1]}P$ .  $\square$

In fact, the argument used to show that  $\text{SBQP} = \text{SBQ}_{[1]}P$  in the proof of Theorem 7 can be extended to prove the following amplification property of  $\text{SBQ}_{[1]}P$ , which is analogous to the cases of  $\text{SBQP}$  and  $\text{SBP}$ .

**Theorem 28.** *For any polynomially bounded function  $p: \mathbb{Z}^+ \rightarrow \mathbb{N}$ , there exists a polynomially bounded function  $q: \mathbb{Z}^+ \rightarrow \mathbb{N}$  such that*

$$\text{SBQ}_{[1]}P \subseteq Q_{[1]}P(2^{-q} \cdot (1 - 2^{-p}), 2^{-q} \cdot 2^{-p}).$$

*Proof.* It suffices to show that for any polynomially bounded function  $p: \mathbb{Z}^+ \rightarrow \mathbb{N}$ , there exists a polynomially bounded function  $q: \mathbb{Z}^+ \rightarrow \mathbb{N}$  such that  $\text{co-SBQP} \subseteq Q_{[1]}P(1 - 2^{-q} \cdot 2^{-p}, 1 - 2^{-q} \cdot (1 - 2^{-p}))$  holds, due to Theorem 7.

Fix a polynomially bounded function  $p: \mathbb{Z}^+ \rightarrow \mathbb{N}$ , and consider any problem  $A = (A_{\text{yes}}, A_{\text{no}})$  is in  $\text{co-SBQP}$ . By the amplification property of  $\text{SBQP}$  presented in Ref. [Kup15], there exists a polynomial-time uniformly generated family  $\{Q_x\}_{x \in \Sigma^*}$  of quantum circuits such that, for every input  $x$ ,  $Q_x$  acts over  $w(|x|)$  qubits, for some polynomially bounded function  $w: \mathbb{Z}^+ \rightarrow \mathbb{N}$ , and the acceptance probability  $p_{\text{acc}}(Q_x, w(|x|))$  of the circuit  $Q_x$  is at least  $1 - 2^{-r} \cdot 2^{-p-1}$  if  $x$  is in  $A_{\text{yes}}$ , while it is at most  $1 - 2^{-r} \cdot (1 - 2^{-p-1})$  if  $x$  is in  $A_{\text{no}}$ , for some polynomially bounded function  $r: \mathbb{Z}^+ \rightarrow \mathbb{N}$ .

Fix an input  $x$ , and consider the quantum circuit  $R_x$  corresponding to the ONE-CLEAN-QUBIT SIMULATION PROCEDURE induced by  $Q_x$  and  $w(|x|)$ . By the properties of  $Q_x$ , Proposition 17 ensures that the acceptance probability  $p_{\text{acc}}(R_x, 1)$  of the one-clean-qubit computation induced by  $R_x$  is at least

$$\begin{aligned} & 1 - 2^{-w(|x|)} \left[ 1 - (1 - 2^{-r(|x|)} \cdot 2^{-p(|x|)-1})^2 \right] \\ & > 1 - 2^{-w(|x|)} [1 - (1 - 2^{-r(|x|)} \cdot 2^{-p(|x|)})] = 1 - 2^{-w(|x|) - r(|x|) - p(|x|)} \end{aligned}$$

if  $x$  is in  $A_{\text{yes}}$ , while it is at most

$$\begin{aligned} & 1 - 2^{-w(|x|)} \left[ 1 - (1 - 2^{-r(|x|)} \cdot (1 - 2^{-p(|x|)-1})) \right] \\ & = 1 - 2^{-w(|x|) - r(|x|)} \cdot (1 - 2^{-p(|x|)-1}) < 1 - 2^{-w(|x|) - r(|x|)} \cdot (1 - 2^{-p(|x|)}) \end{aligned}$$

if  $x$  is in  $A_{\text{no}}$ , which implies that  $A$  is in  $Q_{[1]}P(1 - 2^{-q} \cdot 2^{-p}, 1 - 2^{-q} \cdot (1 - 2^{-p}))$  for  $q = w + r$ , and the claim follows.  $\square$

Now we are ready to prove Theorem 8. More formally, we prove the following statement.

**Theorem 29.** *Suppose that any polynomial-time uniformly generated family of quantum circuits, when used in DQC1 computations, is weakly simulatable with multiplicative error  $c \geq 1$  or exponentially small additive error. Then  $\text{PH} = \text{AM}$ .*

Theorem 29 follows directly from Lemmas 30, 31, and 32 below, combined with the fact that  $\text{AM} \subseteq \text{PH}$ .

**Lemma 30.** *Suppose that any polynomial-time uniformly generated family of quantum circuits, when used in DQC1 computations, is weakly simulatable with multiplicative error  $c \geq 1$  (resp., exponentially small additive error). Then  $\text{NQP} \subseteq \text{NP}$  (resp.,  $\text{NQP} \subseteq \text{SBP}$ ).*



*Proof.* Fix any problem  $A = (A_{\text{yes}}, A_{\text{no}})$  in NQP. By Theorem 7, there exists a polynomial-time uniformly generated family  $\{R_x\}_{x \in \Sigma^*}$  of quantum circuits such that, for every  $x \in \Sigma^*$ , the acceptance probability  $p_{\text{acc}}(R_x, 1)$  of the DQC1 computation induced by  $R_x$  is nonzero if  $x$  is in  $A_{\text{yes}}$ , while it is zero if  $x$  is in  $A_{\text{no}}$ . From the assumption of this lemma, there exists a polynomial-time uniformly generated family  $\{R'_x\}_{x \in \Sigma^*}$  of randomized circuits that weakly simulates  $\{R_x\}_{x \in \Sigma^*}$  with multiplicative error  $c \geq 1$ . Now the definition of weak simulatability with multiplicative error ensures that the probability that the circuit  $R'_x$  outputs 0 (corresponding to acceptance) is nonzero if and only if  $p_{\text{acc}}(R_x, 1)$  is nonzero, which happens only when  $x$  is in  $A_{\text{yes}}$ . This implies that  $A$  is in NP.

In the case where  $\{R'_x\}_{x \in \Sigma^*}$  weakly simulates  $\{R_x\}_{x \in \Sigma^*}$  with exponentially small additive error, the proof uses the fact that the membership of  $A$  in NQP is witnessed by a polynomial-time uniformly generated family  $\{Q_x\}_{x \in \Sigma^*}$  of quantum circuits, where each  $Q_x$  is composed only of Hadamard, Toffoli, and NOT gates (more precisely, each  $Q_x$  is composed only of Hadamard,  $T$ , and CNOT gates, satisfying the definition of NQP in the present paper, but may be assumed to be composed in such a way that  $T$  and CNOT gates are used only for applying Toffoli and NOT transformations). Notice that, for such  $Q_x$ , the acceptance probability, if it is nonzero, must be at least  $2^{-p(|x|)}$  for some polynomially bounded function  $p: \mathbb{Z}^+ \rightarrow \mathbb{N}$ . Hence, from Theorem 7 and its proof, the family  $\{R_x\}_{x \in \Sigma^*}$  of quantum circuits that witnesses the membership of  $A$  in  $\text{NQP}_{[1]} = \text{NQP}$  may be assumed to be such that, for each  $R_x$ , the acceptance probability  $p_{\text{acc}}(R_x, 1)$ , if it is nonzero, must be at least  $2^{-q(|x|)}$  for some polynomially bounded function  $q: \mathbb{Z}^+ \rightarrow \mathbb{N}$ . Furthermore, from the definition of weak simulatability with exponentially small additive error, the family  $\{R'_x\}_{x \in \Sigma^*}$  may be assumed to simulate  $\{R_x\}_{x \in \Sigma^*}$  with additive error at most  $2^{-q(|x|)-2}$ . This implies that the probability that  $R'_x$  outputs 0 is at least  $3 \cdot 2^{-q(|x|)-2}$  if  $x$  is in  $A_{\text{yes}}$ , while it is at most  $2^{-q(|x|)-2}$  if  $x$  is in  $A_{\text{no}}$ , which ensures that  $A$  is in SBP.  $\square$

**Lemma 31.** *If  $\text{NQP} \subseteq \text{NP}$ , then  $\text{PH} \subseteq \text{AM}$ .*

**Lemma 32.** *If  $\text{NQP} \subseteq \text{SBP}$ , then  $\text{PH} \subseteq \text{AM}$ .*

The proof of Lemma 31 requires the notion of the BP operator, whereas the proof of Lemma 32 requires the notion of the  $\widehat{\text{BP}}$  operator, a variant of the BP operator introduced in Ref. [TO92].

For any complexity class  $\mathcal{C}$  of promise problems, a promise problem  $A = (A_{\text{yes}}, A_{\text{no}})$  is in  $\text{BP} \cdot \mathcal{C}$  iff there exist a promise problem  $B = (B_{\text{yes}}, B_{\text{no}})$  in  $\mathcal{C}$  and a polynomially bounded function  $r: \mathbb{Z}^+ \rightarrow \mathbb{N}$  such that, for every  $x$  in  $\Sigma^*$ , it holds that

$$\begin{aligned} x \in A_{\text{yes}} &\implies \left| \{z \in \Sigma^{r(|x|)} : \langle x, z \rangle \in B_{\text{yes}}\} \right| \geq \frac{2}{3} \cdot 2^{r(|x|)} \quad \text{and} \\ x \in A_{\text{no}} &\implies \left| \{z \in \Sigma^{r(|x|)} : \langle x, z \rangle \in B_{\text{no}}\} \right| \geq \frac{2}{3} \cdot 2^{r(|x|)}. \end{aligned}$$

Similarly, for any complexity class  $\mathcal{C}$  of promise problems, a promise problem  $A = (A_{\text{yes}}, A_{\text{no}})$  is in  $\widehat{\text{BP}} \cdot \mathcal{C}$  iff for any polynomially bounded function  $q: \mathbb{Z}^+ \rightarrow \mathbb{N}$ , there exist a promise problem  $B = (B_{\text{yes}}, B_{\text{no}})$  in  $\mathcal{C}$  and a polynomially bounded function  $r: \mathbb{Z}^+ \rightarrow \mathbb{N}$  such that, for every  $x$  in  $\Sigma^*$ , it holds that

$$\begin{aligned} x \in A_{\text{yes}} &\implies \left| \{z \in \Sigma^{r(|x|)} : \langle x, z \rangle \in B_{\text{yes}}\} \right| \geq (1 - 2^{-q(|x|)}) \cdot 2^{r(|x|)} \quad \text{and} \\ x \in A_{\text{no}} &\implies \left| \{z \in \Sigma^{r(|x|)} : \langle x, z \rangle \in B_{\text{no}}\} \right| \geq (1 - 2^{-q(|x|)}) \cdot 2^{r(|x|)}. \end{aligned}$$

It is easy to see that  $\text{AM} = \text{BP} \cdot \text{NP}$ . By the standard error reduction of AM, one can also see that  $\text{AM} = \text{BP} \cdot \text{NP} = \widehat{\text{BP}} \cdot \text{NP}$ .

Now Lemma 31 is proved as follows.

*Proof of Lemma 31.* The claim follows from the following sequence of containments:

$$\text{PH} \subseteq \text{BP} \cdot \text{co-C=P} = \text{BP} \cdot \text{NQP} \subseteq \text{BP} \cdot \text{NP} = \text{AM},$$

where the first inclusion is by Corollary 2.5 of Ref. [TO92] or Corollary 5.2 of Ref. [Tar93], and we have used the fact that  $\text{NQP} = \text{co-C=P}$  [FGHP99], the assumption  $\text{NQP} \subseteq \text{NP}$  of this lemma, and the fact that  $\text{AM} = \widehat{\text{BP}} \cdot \text{NP}$ , respectively.  $\square$

Similarly, Lemma 32 is proved as follows.

*Proof of Lemma 32.* The claim follows from the following sequence of containments:

$$\text{PH} \subseteq \widehat{\text{BP}} \cdot \text{co-C=P} = \widehat{\text{BP}} \cdot \text{NQP} \subseteq \widehat{\text{BP}} \cdot \text{SBP} \subseteq \widehat{\text{BP}} \cdot \text{AM} = \widehat{\text{BP}} \cdot \widehat{\text{BP}} \cdot \text{NP} = \widehat{\text{BP}} \cdot \text{NP} = \text{AM},$$

where the first inclusion is by Corollary 2.5 of Ref. [TO92], the next equality uses the fact that  $\text{NQP} = \text{co-C=P}$  [FGHP99], the next two inclusions are by the assumption  $\text{NQP} \subseteq \text{SBP}$  of this lemma and by the fact that  $\text{SBP} \subseteq \text{AM}$  [BGM06], and the last three equalities use the characterization of  $\text{AM}$  by  $\widehat{\text{BP}} \cdot \text{NP}$  as well as Lemma 2.8 of Ref. [TO92] on the removability of a duplicate  $\widehat{\text{BP}}$  operator.  $\square$

*Remark.* If any polynomial-time uniformly generated family of quantum circuits, when used in DQC1 computations, is weakly simulatable either with multiplicative error  $c \geq 1$  or with exponentially small additive error, then the inclusion  $\text{SBQP} \subseteq \text{SBP}$  can also be proved, by using the fact that  $\text{SBQP} = \text{SBQ}_{[1]} \text{P}$  proved in Theorem 7 as well as the amplification property of  $\text{SBQP}$  (or the amplification property of  $\text{SBQ}_{[1]} \text{P}$  proved in Theorem 28). This can also prove Theorem 8 (more formally, Theorem 29), the hardness result on such weak simulatability of DQC1 computations, as the collapse of  $\text{PH}$  to  $\text{AM}$  also follows from the assumption  $\text{SBQP} \subseteq \text{SBP}$ , due to the inclusion  $\text{NQP} \subseteq \text{SBQP}$ .

Finally, the argument based on  $\text{NQP}$  and  $\text{SBQP}$  used to prove Theorem 8 (more formally, Theorem 29) in this section can also be used to show the hardness of weak classical simulations of other quantum computing models. In particular, it can replace the existing argument based on  $\text{PostBQP}$ , which was developed in Ref. [BJS11] and has appeared frequently in the literature [AA13, JVDN14, MFF14, TYT14, Bro15, TTYT15]. This also weakens the complexity assumption necessary to prove the hardness results for such models, including the IQP model [BJS11] and the Boson sampling [AA13] (the polynomial-time hierarchy now collapses to the second level, rather than the third level when using  $\text{PostBQP}$ ). Moreover, the hardness results for such models now hold for any constant multiplicative error  $c \geq 1$ , rather than only for  $c$  satisfying  $1 \leq c < \sqrt{2}$  as in Refs. [BJS11, MFF14].

## 9 Conclusion

This paper has developed several error-reduction methods for quantum computation with few clean qubits, which simultaneously reduce the number of necessary clean qubits to just one or two. Using such possibilities of error-reduction, this paper has shown that the  $\text{TRACE ESTIMATION}$  problem is complete for  $\text{BQ}_{\log} \text{P}$  and  $\text{BQ}_{[1]} \text{P}$ . One of the technical tools has also been used to show the hardness of weak classical simulations of DQC1 computations. A few open problems are listed below concerning the power of quantum computation with few clean qubits:

- In the case of one-sided bounded error, can any quantum computation with logarithmically many clean qubits be made to have exponentially small one-sided error by just using one clean qubit, rather than two? A similar question may be asked even in the case of two-sided bounded error whether both completeness and soundness errors can be made exponentially small simultaneously by just using one clean qubit.
- In the two-sided error case, is error-reduction possible even when a starting quantum computation with few clean qubits has only an inverse-polynomial gap between completeness and soundness?
- Are DQC1 computations provable to be hard to classically simulate under a more desirable notion of simulatability, like those discussed in Refs. [BJS11, AA13]? Such results are known for other computation models like Boson sampling by assuming hardness of some computational problems [AA13, BMS15, FU15].

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